Abstract

In this paper we introduce a new analytical approach to pricing American options. We use an explicit and intuitively appealing proxy for the exercise rule, and derive tractable pricing formulas using a short-maturity asymptotic expansion. Numerical experiments show that the analytical approximation is fast and accurate for options with time-to-maturity up to several years under typical model parameters. In the Black-Scholes case the approximation is as precise as a binomial tree with several hundred steps with the speed of a 25-step tree. The main advantage of our approach lies in its application to a three-factor model with stochastic volatility and stochastic interest rates while keeping computational time low.

Key words: American options, stochastic volatility, stochastic interest rates, asymptotic approximation.

JEL Classification: G12.

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1 Introduction

The valuation of American options is a challenging task even under the Black-Scholes model (see Detemple (2005) for an extensive review). There exist a number of semi-analytical approximations for the American option prices that partially solve this problem (Barone-Adesi and Whaley (1987), Broadie and Detemple (1996), Bunch and Johnson (2001), Ju (1998)). These approaches are fast and accurate, but they cannot be easily extended beyond the Black-Scholes model.

It has been firmly established that the Black-Scholes model is not consistent with quoted option prices. The literature advocates the introduction of stochastic volatility to reproduce the implied volatility smile observed in the market. The introduction of a second stochastic factor complicates enormously the pricing of American options. Presently, this can only be done by means of numerical schemes, which involve solving integral equations (Kim (1990), Huang, Subrahmanyam and Yu (1996), Sullivan (2000), Detemple and Tian (2002)), performing Monte Carlo simulations (Broadie and Glasserman (1997), Longstaef and Schwartz (2001), Rogers (2002), Haugh and Kogan (2004)), or discretizing the partial differential equation (Brennan and Schwartz (1977), Clarke and Parrott (1999), Ito and Toivanen (2007)).

The early exercise premium of the American put option depends on the cost of carry determined by interest rates. Consequently, the volatility of interest rates does affect the decision to exercise this option at any point in time. This fact is recognized in the literature dealing with models with stochastic interest rates (Detemple and Tian (2002), Ho, Stapleton and Subrahmanyam (1997), Merkved and Vorst (2001)). This literature, however, considers only two-factor extensions of the Black-Scholes model assuming that the volatility of the underlying is constant. Up to now, there exists no feasible methodology for pricing American options when volatility and interest rates are stochastic.

In this paper we propose a new analytical approach that is both computationally tractable and general enough to be successfully applied to a three-factor model. Our approach is based on the idea of substituting the optimal exercise rule with a simple (suboptimal) exercise rule for which an approximate solution is easy to find and fast to compute. Similar ideas have already been explored in the literature in the context of the Black-Scholes model (Broadie and Detemple (1996), Carr (1998), Ju (1998)). Our proxy rule is to exercise the option as soon as its moneyness measured in standard deviations reaches some predefined level. The price of such an option appears to have a regular asymptotic behavior near maturity with an asymptotic expansion available in a closed form for a broad class of models. The American option price is then approximated by the maximum over these option prices. We provide several numerical experiments and comparisons showing that our method performs well with respect to computational time and accuracy.

The rest of the paper is organized as follows. In Section 2 we describe our approach in the context of the Black-Scholes model. We provide a motivation for our approach, discuss intuitively
its main features, and compare it with other available methods. We also study numerically the accuracy of approximation of the option delta. In Section 3 we generalize it to a three-factor model with stochastic volatility and stochastic interest rates. We run several numerical experiments to show that our approach is competitive with existing methods. The section is concluded by a numerical analysis under an affine model with both stochastic volatility and stochastic interest rates. In particular, we quantify the effect of the two additional stochastic factors on the early exercise premium for different parameter values. Section 4 concludes the paper. Technical proofs and results are gathered in Appendices. All the Matlab codes used in this paper are available on request from the authors.

2 Black-Scholes model

In this section we consider the Black-Scholes model where the stock price follows a log-normal diffusion and develop analytical approximation for the American put. This section is aimed at presenting intuitively our approach in a simple setting before looking at richer settings.

2.1 Short-maturity asymptotics for American option prices

An American put option with strike price $K$ and maturity date $T$ is a derivative that gives the owner the right to receive $\max(K - S_t, 0)$ at any point in time $t < T$. Under the Black-Scholes model the price $P(S, t)$ of this option satisfies the partial differential equation (PDE):

$$P_t + (r - \delta)SP_S + \frac{1}{2}\sigma^2 S^2 P_{SS} = 0,$$

with boundary conditions:

$$P(\infty, t) = 0,$$

$$P(S, T) = \max(K - S, 0),$$

$$P(\mathcal{S}(T-t), t) = \max(K - \mathcal{S}(T-t), 0),$$

$$P_S(\mathcal{S}(T-t), t) = -1.$$  

Here subscripts denote differentiation with respect to time $t$ and stock price $S$; $r$ and $\delta$ are the interest rate and the dividend yield; $\mathcal{S}(\tau)$ is the early exercise price, which depends on the option time-to-maturity $\tau = T - t$. The last boundary condition (2) is the so called "smooth pasting" condition. Note that the European put also satisfies PDE (1) with the boundary conditions with zero early exercise price $\mathcal{S}(\tau) \equiv 0$. The unique solution for the American option price is then determined by additional condition $P(S, t) \geq \max(K - S, 0)$. 

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Solving PDE (1) given the five conditions is a non-trivial task. The only known analytical solution to this problem\(^2\) is found by Zhu (2006) in the form of a Taylor series expansion. While the emphasis of this solution is to show the existence of an analytical exact solution for this quite intricate mathematical problem, it does not have a clear advantage over some fast numerical schemes as far as the realization of the numerical values of the solution is concerned. The convergence is rather slow, and the expansion terms are given recursively by a lengthy analytical formula.

A number of approximation methods exist in the literature. In particular, the behavior of \( S(\tau) \) near maturity (small \( \tau \)) has attracted lots of attention as a promising way to derive an analytical formula (Allobaidi and Mallier (2001), Barles et al. (1995), Chevalier (2005), Dewynne et al. (1993), Evans, Keller and Kuske (2002), Goodmand and Ostrov (2002), Lamberton and Villeneuve (2003); see also Levendorski (2007) and references therein for further general results). However, this approach has not produced any approximation that would be sufficiently accurate under realistic model parameters (see the numerical examples in Mallier (2002)). Although a high order short-maturity asymptotic approximation of \( S(\tau) \) is available in an analytical form (see Allobaidi and Mallier (2004)), accuracy is still an issue.

Let us now show why a direct short-maturity analysis does not yield an applicable formula for American options. This will motivate the introduction of a new method of option pricing. Consider the choice to exercise a put option now or wait till maturity. For illustration we consider the case of zero dividend. If the option is exercised now then the option holder receives \( K S_t \) (provided that \( K > S_t \)). The expected payoff of the option at maturity is equal to the European put price. From the put-call parity it can be written as:

\[
Ke^{-r\tau} - S_t + C^E,
\]

where \( C^E \) is the European call price. If \( K \) is sufficiently greater than \( S_t \) (option is said to be deep in-the-money) then \( C^E \) is negligibly small. In this case the immediate exercise of the option yields a benefit:

\[
K(1 - e^{-r\tau}) > 0,
\]

equal to the interest rate income on \( K \).

To measure the moneyness of the put option we introduce a convenient parameterization, denoting:

\[
\theta = \frac{\ln (K/S)}{\sigma \sqrt{\tau}}.
\]

This ratio is called normalized moneyness, and is frequently used in the empirical literature on option pricing (see e.g. Bates (2000), Carr and Wu (2003)). It measures the distance between log

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\(^2\) Except, of course, the trivial case when there is no early exercise premium.
of stock price and log of strike price in terms of standard deviations\(^3\). The value of \(\theta\) is a good indicator to decide whether or not to exercise the option. Indeed, if \(\theta\) is above 1.65 then there is only a 5\% chance that the put option ends out-of-the-money at maturity. Consequently, the call option price \(C^E\) in (3) can be neglected, and the option holder should exercise the put right away.

Although this reasoning seems to be plausible, it appears to be not entirely correct. Indeed, the short-maturity asymptotics of \(\overline{S}(\tau)\) (see e.g. Barles et al. (1995)) implies \(\ln \left(\frac{\overline{S}(\tau)}{K}\right) \sim \sqrt{-\tau \ln(\tau)}\), meaning that

\[
\overline{\theta}(\tau) \equiv \frac{\ln \left(\frac{K}{\overline{S}(\tau)}\right)}{\sigma \sqrt{\tau}} \sim \sqrt{-\ln(\tau)}.
\]

As we observe from (6), when \(\tau\) is very small, no matter how large is the gap between the stock price and the strike price, it is still not optimal to exercise the option early. This obviously does not agree with the intuitive exercise rule described above. The explanation of this apparent contradiction is that the call option price \(C^E\) is \(O(\sqrt{\tau})\) for given fixed \(\theta\) (see Medvedev and Scaillet (2007) and (18)). Consequently, when \(\tau \to 0\) the early exercise benefit (4) is only second order relevant, and the American put turns into a European put with \(\overline{S} = 0\) and \(\overline{\theta} = \infty\).

For realistic time-to-maturity, the early exercise premium is not negligible. Consequently, a direct short-maturity asymptotic analysis is not able to deliver a good approximation. However, it is still possible to rely on a short-maturity asymptotic analysis if we modify the initial problem. Ideally, the solution to the modified problem should be very close to the American option price, and have regular asymptotics when time-to-maturity goes to zero. This can be done by imposing an intuitive exercise rule based on normalized moneyness \(\theta\).

### 2.2 Modified problem

Let us consider a modification of the problem (1) with "smooth pasting" condition (2) substituted with an explicit exercise rule. The new problem is defined through the same PDE

\[
P_t + (r - \delta)SP_S + \frac{1}{2} \sigma^2 S^2 P_{SS} = 0,
\]

with boundary conditions:

\[
P(\infty, t) = 0,
\]

\[
P(S, T) = \max(K - S, 0),
\]

\[
P(\overline{S}(T - t), t) = \max \left( K - \overline{S}(T - t), 0 \right),
\]

\[
\text{\footnotesize{3}\text{Strictly speaking, we should take the ratio of the strike price to the forward price to take into account the drift in log } S. \text{ The two definitions, however, are equivalent when time-to-maturity goes to zero.}}
\]

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where \( \overline{S}(T-t) \) satisfies \( \frac{\ln \left( \frac{K}{\overline{S}(T-t)} \right)}{\sigma \sqrt{T-t}} = y \).

The unique solution to this problem is the price of an option that is exercised as soon as the normalized moneyness reaches the level of \( y \). As we have already noted this proxy for the optimal exercise rule is intuitively appealing, and we expect the solution to the modified problem to be close to the true American option price.

To derive proper asymptotic expansions, we rewrite PDE (7) in terms of \((\theta, \tau)\) instead of \((S,t)\). Using definition (5) of \( \theta \) and denoting \( P(\theta, \tau) \equiv P \left( Ke^{-\sigma \theta \sqrt{\tau}}, T - \tau \right) \), we make the following substitutions in (7): \( P_t = -P_{\tau} + \frac{\theta}{2\tau} P_{\theta} \), \( P_S = -\frac{1}{\sigma S \sqrt{\tau}} P_{\theta} \), and \( P_{SS} = P_{\theta \theta} - \frac{1}{\sigma^2 S^2 \tau} + \frac{1}{\sigma S \sqrt{\tau}} P_{\theta} \). After simplification we obtain:

\[
\theta P_{\theta} + P_{\theta \theta} + \frac{1}{\sigma} \left[ \sigma^2 + 2(\delta - \tau) \right] P_{\theta \sqrt{\tau}} - (P_{\tau} + \tau P) \tau = 0. \tag{11}
\]

As we will see in the next section, there exists a unique solution to (11), which satisfies the boundary conditions (8) and (10) in the form:

\[
P(-\infty, \tau) = 0, \tag{12}
\]

\[
P(y, \tau) = K \max \left( 1 - e^{-\sigma y \sqrt{\tau}}, 0 \right) = K \left( 1 - e^{-\sigma y \sqrt{\tau}} \right), \tag{13}
\]

and which has regular asymptotics near maturity of the form:

\[
P(\theta, \tau) = \sum_{n=1}^{\infty} P_n(\theta) \tau^{\frac{n}{2}}. \tag{14}
\]

The third condition (9) is hidden in the form of the asymptotics (14). Indeed, when \( \tau = 0 \) and \( \theta \) is held fixed we have \( S = 0 \) and \( \max(K - S, 0) = 0 \). Hence we have \( P(\theta, 0) = 0 \), which is implicit in (14).

Let us denote the solution to the modified problem (11) with conditions (12), (13), and (14) by \( P(\theta, \tau; y) \). The American put price can be approximated from below by:

\[
P(\theta, \tau) \simeq P(\theta, \tau; y^*(\theta)) = \max \{ P(\theta, \tau; y) : y > 0 \}. \tag{15}
\]

From the discussion in the previous section we expect that the optimal \( y^* \) should be somewhere between 1 and 2 in case \( \delta = 0 \). Besides, when \( y \) goes to infinity, the solution to the modified problem converges to the European option price denoted by \( P(\theta, \tau; \infty) \). Note also that if \( r = 0 \) and \( \delta > 0 \) then it is not optimal to exercise the American put before its expiration, and \( y^* = \infty \). In Figure 1(a) and 1(b) we confirm these statements using two numerical examples with \( r = 0.05, \delta = 0 \) and \( r = 0, \delta = 0.05 \). In both figures we plot the solution \( P(0, 1; y) \) to the modified problem corresponding to an at-the-money put option \((\theta = 0)\) with 1 year to maturity \((\tau = 1)\).
2.3 Asymptotic expansion

Let us introduce the following notations: 
\[ \Phi(\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\theta} e^{-\frac{s^2}{2}} ds, \quad \phi(\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}}. \]

In the next Proposition we describe the series representation of the general solution to (11) without boundary condition (13). Then we show how a unique solution is determined by requiring (13).

**Proposition 1** Consider partial differential equation (11) with boundary condition (12) and the regular asymptotic expansion (14) in the vicinity of \((0,0)\). For any solution to this problem there exist constants \(C_1, C_2, \ldots, C_n\) such that for each \(n\):

\[
P_n(\theta) = C_n \left[ p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta) \right] + p_n^1(\theta)\Phi(\theta) + q_n^1\phi(\theta),
\]

with \(p_n^0, p_n^1, q_n^0, q_n^1\) being polynomials in \(\theta\) of the form:

\[
p_n^0(\theta) = \theta^n + \pi_{n1}\theta^{n-2} + \pi_{n2}\theta^{n-4} + \ldots,
q_n^0(\theta) = \theta^{n-1} + \pi_{n1}\theta^{n-3} + \pi_{n2}\theta^{n-5} + \ldots,
\]

\[
p_n^1(\theta) = \pi_{n1}\theta^{n-2} + \pi_{n2}\theta^{n-4} + \ldots,
q_n^1(\theta) = \pi_{n1}\theta^{n-3} + \pi_{n2}\theta^{n-5} + \ldots
\]

with coefficients depending on model parameters and \(C_1, C_2, \ldots, C_{n-1}\).

**Proof.** See Appendix A. \(\blacksquare\)

Proposition 1 describes the form of the asymptotic expansion of the general solution (14) with appropriate behavior at infinity (12). To determine a unique \(N\)th order expansion, we need to select \(N\) constants \(C_n, n = 1, \ldots, N\). Let us show how to do this using a second order expansion of equation (14) as an illustration. Using Proposition 1 we find by substitution:

\[
P(\theta, \tau) = C_1 \left[ \theta\Phi(\theta) + \phi(\theta) \right] \sqrt{\tau}
\]

\[
+ \left[ C_2 \left( (\theta^2 + 1)\Phi(\theta) + \theta\phi(\theta) \right) + \frac{C_1}{2\sigma} \left( \sigma^2 - 2r + 2\delta \right) \Phi(\theta) \right] \tau + O(\tau^{1/2}).
\]  

(16)

The coefficients \(C_1\) and \(C_2\) are uniquely determined by imposing the early exercise condition (13). The short-maturity expansion of the payoff function is:

\[
P(y, \tau; y) = K(1 - \exp(-\sigma y\sqrt{\tau}))
\]

\[
= K \left( \sigma y\sqrt{\tau} - \frac{\sigma^2 y^2}{2} \tau \right) + O(\tau^{1/2}).
\]  

(17)

Equalizing expansion (16) at \(\theta = y\) to expansion (17), we obtain a system of two equations which can be solved recursively. The expressions for these coefficients can be found in Appendix B, where
we present the short-maturity expansion of $P(\theta, \tau; y)$ up to the 4th order. Recall that the European put price is obtained by setting $y = \infty$. We can check that when $y \to \infty$ we have $C_1(\infty) = \sigma K$, $C_2(\infty) = -\frac{K\sigma^2}{2}$, and:

$$
P(\theta, \tau; \infty) = \sigma K (\theta \Phi(\theta) + \phi(\theta))\sqrt{\tau} - K \left[ \frac{\sigma^2}{2} (\theta^2 \Phi(\theta) + \theta \phi(\theta)) + r \Phi(\theta) \right] \tau + O(\tau). \quad (18)
$$

This is exactly the asymptotics of the European put implied by the put-call parity and results in Medvedev and Scaillet (2007) for the European call.

2.4 Early exercise

In the previous section we described a method to approximate the American put price. However, we did not discuss how to decide on the early exercise of the option. The decision to exercise the option should be based on comparison of $y$ and the optimal level of moneyness $y^*(\theta)$. If $\theta > y^*(\theta)$ then the option should be exercised. Formally, the best approximation for the early exercise level of $\theta$ can be found from the fixed point problem:

$$
\bar{\theta} = y^*(\bar{\theta}). \quad (19)
$$

2.5 Performance of the approximation

In this section we perform several numerical experiments to study the accuracy of the approximation of the American put option introduced in the previous section. The approximation error has two possible sources: the asymptotic expansion and the suboptimal exercise rule. Hereafter we find that the convergence of the asymptotic expansion is extremely fast even at long maturities, meaning that the major error comes from the suboptimal exercise rule. This approximation error also appears to be small relative to the option price.

2.5.1 Convergence of the asymptotic expansion

To illustrate the speed of convergence of the asymptotic expansion we consider three one-year put options with moneyness $K/S = 0.8, 1.0$ and $1.2$. The interest rate, dividend yield and the volatility are $r = 0.05$, $\delta = 0$, $\sigma = 0.2$. The optimal $y^*$ is found using a simple search algorithm. We start with $y = 1.5$ and then move in the direction of increasing price with a step of 0.1. When this preliminary search is terminated, we refine the search with a smaller step of 0.01. This procedure allows us to find $y^*$ with a precision of 0.01. In Figure 2 we plot the approximation errors relative to the option price as a function of the number of expansion terms in (14). Here the true price is computed using the binomial tree method with 1000 time steps.
Observe that the convergence of the short-maturity asymptotic expansion is very fast. The 4th order expansion seems to be sufficient with a relative error below 0.5%. These results are remarkable since we consider relatively long matured options. The convergence of the expansion and the approximation accuracy are even better for options with shorter maturities. The closed form approximation based on the 4th order expansion is given in Appendix B.

2.5.2 Approximation of the option delta

Option Greeks can be directly computed by taking corresponding derivatives of the approximation formula and evaluating them at $y^*$. In practical applications it is important that not only option prices but also their Greeks are computed accurately. Here we consider the accuracy of the approximation of the option delta, which is the price derivative of the option price.

Recall that we have modified the original problem for the American put by dropping the "smooth pasting" condition. This condition states that the option delta at the early exercise should be always equal to -1. Let us study the extent to which this condition is violated by our approximation. To solve for the early exercise level of $\theta$ we use the following modification of (19):

$$\bar{y} \approx \min \{ \theta \in \Theta : \theta \geq y^*(\theta) \},$$

where $\Theta = \{0.01n, n = 1, 2, \ldots\}$ and $y^*$ is computed using the same search algorithm as in the previous section. This approach allows solving (19) with a precision of 0.01.

In Figure 3 we plot the option deltas evaluated at $\bar{y}$ for different time-to-maturity and volatility. The errors are negligible not exceeding 0.005.

2.5.3 Comparison with existing methods

In this section we compare our approach with other analytical approximations developed for the Black-Scholes model. We perform the analysis using model parameters chosen in the corresponding paper.

Broadie and Detemple (1996) suggest simple lower and upper bounds on the American call price. The lower bound is computed as the maximum over prices of call options that are exercised as the price level reaches some critical value (capped call options). The difference between our approach and that of Broadie and Detemple (1996) is that we use this rule for the normalized moneyness rather than the price of the underlying. We believe that expressing the suboptimal exercise rule in terms of normalized moneyness is more appealing, and we expect our approach to be more accurate. Although the price of the capped option in Broadie and Detemple (1996) admits an exact analytical expression, our approximation is given by an asymptotic expansion with a fast convergence (see the previous section).
Table 1 reports the lower and upper bounds on American call prices from Broadie and Detemple (1996) (Tables 1 and 2), along with our results. To gauge the early exercise premium we also give European option prices. We compute American call prices using the put-call symmetry (see Broadie and Detemple (2004)). The call option price is equal to the put option price with $S$ replaced by $K$ and vice-versa, and $r$ replaced by $\delta$, and vice-versa. In Table 1 we illustrate convergence by computing option values using asymptotic expansions of different orders. Broadie and Detemple (1996) also develop two alternative approximations LBA and LUBA. LBA is constructed by adjusting the lower bound approximation by a multiplicator estimated numerically. LUBA is an average between the lower and upper bounds with numerically estimated weights. We do not report them since the comparison of our lower bound approximation with them would be inadequate. For proper comparison we should perform a similar numerical adjustment, which is beyond the scope of this paper. It also should be noted that such an ad hoc approximation may result in inaccurate values of option Greeks.

The first part of Table 1 reports option values corresponding to time-to-maturity equal to 6 months. Here the convergence of the asymptotic expansion is sufficiently fast and the 4th order expansion is largely sufficient. The accuracy of our approximation is comparable with that of a 300-step binomial tree and is clearly superior to both bounds of Broadie and Detemple (1996). The relative error does not exceed 0.2%, which is more than sufficient for applications. In the second part of Table 1 we compare different approximations of option prices with long time-to-maturity (3 years). The convergence of the series here is much slower. This is expected given that we rely on a short-maturity expansion. Accuracy is still reasonably good even if we limit ourselves, for example, to a 5th order expansion with a relative error not exceeding 0.5%. In most cases our approximation based on the 5th order expansion has a better accuracy than the lower bound of Broadie and Detemple (1996). Our option values appear to be less accurate only in the last combination of option parameters, where the American call price and the European call price are equal.

With respect to the computational time our method is roughly similar to the lower bound approximation of Broadie and Detemple (1996). Both methods involve similar maximization procedure, and formulas have comparable complexities. To give an idea of the computational speed, a Matlab code requires only 0.015 seconds to compute an approximation using a 4th order expansion. This is comparable to a 25-step binomial tree on the same computer. Note that Broadie and Detemple (1996) approach may still be preferable for pricing long maturity options. In this case, our approximation requires higher order expansion, which increases the computational time.

Bunch and Johnson (2000) propose an alternative fast method for American option pricing based on an analytical approximation of the early exercise boundary. Table 2 reports option values from Bunch and Johnson (2000) (Table II) and our results. We compute our approximation using a 4th order asymptotic expansion (see Appendix B) and the search algorithm described above. The accuracy of our approximation and the speed of computations are comparable with that of Bunch
and Johnson (2000).

3 Stochastic volatility and stochastic interest rates

The true power of the method introduced in the previous sections is in its broad applicability: it can be extended to a more general model with stochastic volatility and stochastic interest rates. Let us consider the following risk-neutral dynamics:

\[
\begin{align*}
    dS_t &= (r_t - \delta)S_t dt + \sigma_t S_t dW^{(1)}_t, \\
    d\sigma_t &= a(\sigma_t) dt + b(\sigma_t) dW^{(2)}_t, \\
    dr_t &= \alpha(r_t, t) dt + \beta(r_t) dW^{(3)}_t,
\end{align*}
\]

with \(dW^{(i)}_t dW^{(j)}_t = \rho_{ij}\). Model (20) nests most of the models used in applications.

3.1 Modified problem and its solution

We proceed in the same manner as previously. The PDE for the put option price \(P(S, \sigma, r, t)\) is:

\[
0 = P_t + P_S S(r - \delta) + P_{\sigma} \sigma(\sigma) + P_r r(r, t) + \frac{1}{2} P_{SSS} S^2 \sigma^2 \\
    + \frac{1}{2} P_{\sigma\sigma} \sigma^2(\sigma) + \frac{1}{2} P_{rr} \sigma^2(r) + P_{S\sigma} \sigma \beta(\sigma) \rho_{12} \\
    + P_{Sr} \sigma \beta(\sigma) \rho_{13} + P_{\sigma r} \beta(\sigma) \beta(\sigma) \rho_{23} - \tau P,
\]

with the boundary conditions as in (1).

Let us make the change of variables from \((S, t)\) to \(\theta = \frac{\log(K/S)}{\sigma \sqrt{T - t}}\) and \(\tau = T - t\) and make use of the following relationships:

\[
\begin{align*}
    P_S &= -\frac{1}{\sigma S \sqrt{\tau}} P_\theta, \\
    P_{SS} &= \frac{1}{\sigma S^2 \sqrt{\tau}} P_{\theta \theta} + \frac{1}{\sigma^2 S \sqrt{\tau}} P_\theta, \\
    P_t &= \frac{1}{2 \tau} P_\theta - P_\tau, \\
    P_\sigma &= P_\sigma - \frac{\theta}{\sigma} P_\theta, \\
    P_{\sigma \sigma} &= -\frac{1}{\sigma S \sqrt{\tau}} P_{\theta \theta} + \frac{1}{\sigma^2 S \sqrt{\tau}} P_\theta + \frac{\theta}{\sigma^2 S \sqrt{\tau}} P_{\theta \theta}, \\
    P_{\sigma r} &= P_{\sigma r} - \frac{2 \theta}{\sigma} P_{\sigma \theta} + \frac{2 \theta}{\sigma^2} P_\theta + \frac{\theta^2}{\sigma^2} P_{\theta \theta}, \\
    P_r &= P_r, \\
    P_S r &= -\frac{1}{\sigma S \sqrt{\tau}} P_{\theta r}, \\
    P_{rr} &= P_{rr},
\end{align*}
\]
\[ \alpha(r, t) = \alpha(r, T - \tau) = \sum_{i=0}^{\infty} \alpha_i(r) r^i. \] (22)

This allows us to transform (21) into:

\[
0 = \theta a P_{\theta} + \frac{1}{2} P_{\theta \theta} - \tau P_{r} + \sqrt{\tau} \left[ \frac{1}{\sigma} P_{\sigma} (\sigma^2/2 - r + \delta) + b P_{12} \left( -P_{\sigma \theta} + \frac{1}{\sigma} P_{\theta} + \frac{\theta}{\sigma} P_{\theta \theta} \right) \right. \\
- \beta P_{13} P_{\theta r} + b \beta P_{23} \left( P_{\sigma r} - \frac{\theta}{\sigma} P_{\theta r} \right) + \tau \left[ a \left( P_{\sigma} - \frac{\theta}{\sigma} P_{\theta} \right) \right. \\
\left. + \frac{b^2}{2} \left( P_{\sigma \sigma} - \frac{2 \theta}{\sigma^2} P_{\sigma \theta} + \frac{2 \theta}{\sigma^2} P_{\theta} + \frac{\theta^2}{\sigma^2} P_{\theta \theta} \right) + \frac{1}{2} P_{rr} \beta^2 - \tau P \right] + P_{r} \sum_{i=0}^{\infty} \alpha_i(r) r^{i+1}. \] (23)

Further, let us take an expansion of the option price near maturity of the form:

\[ P(r, \tau, \sigma) = P_{1} \sqrt{\tau} + P_{2} \tau + P_{3} \sigma \sqrt{\tau} + \ldots \]

Substituting this into (23), we obtain the following PDE for \( P_n \) (\( n > 0 \)):

\[
0 = P_{n \theta \theta} + \theta P_{n \theta} - n P_{n} + \frac{1}{\sigma} P_{n - 1 \theta} (\sigma^2 - 2r + 2\delta) \\
+ 2b P_{12} \left( -P_{n - 1 \sigma \theta} + \frac{1}{\sigma} P_{n - 1 \theta} + \frac{\theta}{\sigma} P_{n - 2 \theta} \right) \\
- 2\beta P_{13} P_{n - 2 \theta \sigma} + 2b \beta P_{23} \left( P_{n - 1 \sigma r} - \frac{\theta}{\sigma} P_{n - 1 \sigma r} \right) \\
+ 2a \left( P_{n - 2 \sigma} - \frac{\theta}{\sigma} P_{n - 2 \theta} \right) + 2 \sum_{i=0}^{n-2} \alpha_i(r) P_{(n-2-2i)r} + b^2 \left( P_{n - 2 \sigma \sigma} - \frac{2 \theta}{\sigma} P_{n - 2 \sigma \theta} \right. \\
\left. + \frac{2 \theta}{\sigma^2} P_{n - 2 \theta \sigma} + \frac{\theta^2}{\sigma^2} P_{n - 2 \theta \theta} \right) + P_{n - 2rr} \beta^2 - 2\tau P_{n - 2}, \] (24)

with \( P_{-1} = P_0 = 0 \).

Proposition 2 describes the solution to (24) in the form of expansion (14) having appropriate behavior at infinity. We do not provide the proof of the proposition, which is lengthy but straightforward. The result can be verified by a direct substitution of the solution with unknown coefficients in (24).

**Proposition 2** Consider partial differential equation (24) with \( \rho_{23} = 0 \), the boundary condition:

\[ P(-\infty, \sigma, r, t) = 0, \]
and regular asymptotic expansion:

\[ P(\theta, \sigma, r, \tau) = \sum_{n=1}^{\infty} P_n(\theta, \sigma, r) r^\frac{\tau}{2}, \]

with \((\theta, \tau)\) in the vicinity of \((0, 0)\). For any solution to this problem there exist functions \(C_1(\sigma, r), C_2(\sigma, r), \ldots\) such that for each \(n\):

\[
P_n(\theta, \sigma, r) = C_n(\sigma, r) \left[ p_n^0(\theta, \sigma, r) \Phi(\theta) + q_n^0(\theta, \sigma, r) \phi(\theta) \right] + p_n^1(\theta, \sigma, r) \Phi(\theta) + q_n^1(\theta, \sigma, r) \phi(\theta),
\]

with \(p_n^0, p_n^1, q_n^0, q_n^1\) being polynomials in \(\theta\) of the form:

\[
p_n^0(\theta, \sigma, r) = \theta^n + \pi_{1n}^0(\sigma, r) \theta^{n-2} + \pi_{2n}^0(\sigma, r) \theta^{n-4} + \ldots,
\]

\[
q_n^0(\theta, \sigma, r) = \theta^{n-1} + \pi_{1n}^0(\sigma, r) \theta^{n-3} + \pi_{2n}^0(\sigma, r) \theta^{n-5} + \ldots,
\]

\[
p_n^1(\theta, \sigma, r) = \pi_{1n}^1(\sigma, r) \theta^{n-2} + \pi_{2n}^1(\sigma, r) \theta^{n-4} + \ldots,
\]

\[
q_n^1(\theta, \sigma, r) = \pi_{1n}^1(\sigma, r) \theta^{3n-5} + \pi_{2n}^1(\sigma, r) \theta^{3n-7} + \ldots,
\]

with coefficients depending on model parameters and \(C_1(\sigma, r), C_2(\sigma, r), \ldots, C_{n-1}(\sigma, r)\).

In Proposition 2 we assume that the spot interest rate is not correlated with the volatility of the underlying. It is probably possible to derive the form of the solution for the general problem with arbitrary correlations. However, the formula will be much more complicated, which is not justified from a practical point of view. It is though important that we are able to incorporate non-zero correlations between the volatility and the price, and between interest rate and the price. As we will see in the numerical analysis below these correlations magnify the effect of stochastic factors on the early exercise premium.

To derive the approximation for the American put we proceed in the same manner as in the Black-Scholes case. We impose the early exercise condition (13) to solve for coefficients \(C_i(\sigma, r), i = 1, 2, \ldots, n, \ldots\) The recursive formulas for these coefficients can be easily obtained from a symbolic calculus software and copy-pasted in a code for option pricing. The American put price is approximately given by

\[
P(\theta, \tau) \simeq P_1(\theta, \tau; y^*(\theta)) \sqrt{\tau} + P_2(\theta, \tau; y^*(\theta)) \tau + \ldots, \tag{25}
\]

where
\[ y^*(\theta) = \arg \max_y \{ P(\theta, \tau; y); y > 0 \}. \] (26)

### 3.2 Approximation of the price vs. approximation of the early exercise premium

The American put option price is the sum of the European put price and the early exercise premium. The approximation error can be decomposed in a similar way. If two errors are relatively independent then the accuracy of our approximation can be improved in models where European put option prices are available in a closed form. A well-known example is the class of affine multifactor models, where European options can be valued quickly and accurately via the inverse Fourier transform (Duffie et al. (2000)).

Recall that the European put price is the solution to the modified problem with \( \overline{\theta} = \infty \). Consequently, the early exercise premium is approximately given by \( P(\theta, \tau; y^*) - P(\theta, \tau; \infty) \), where \( y^* \) is determined by (26). Now suppose that the price of the European put is available. Then the American put price can be approximately given by:

\[
P(\theta, \tau) \simeq P(\theta, \tau; \infty) + [P_1(\theta; y^*(\theta)) - P_1(\theta; \infty)] \sqrt{\tau} + [P_2(\theta; y^*(\theta)) - P_2(\theta; \infty)] \tau + ... \tag{27}
\]

To distinguish between the two approximation we refer to (25) as Approximation 1 and to (27) as Approximation 2.

Let us use a numerical example to check if Approximation 2 performs better indeed. We consider again three one-year put options with moneyness \( K/S = 0.8, 1.0 \) and 1.2, and assume zero dividend yield and constant interest rate \( r = 0.05 \). The volatility is stochastic with the risk-neutral dynamics of the variance \( v_t = \sigma_t^2 \) given by the square-root process:

\[
dv_t = 2(0.04 - v_t)dt + 0.2\sqrt{v_t}dW_t^{(2)} \quad \text{with} \quad \rho_{12} = -0.5.
\]

This model belongs to the affine class and has been first proposed by Heston (1993). In Figure 4 we plot two graphs showing relative errors of Approximations 1 and 2 as a function of the number of terms in the short-maturity expansion. First, the convergence of Approximation 1 under stochastic volatility is much slower than in the Black-Scholes model (compare to Figure 2). Now it requires at least 6 expansion terms to obtain a good approximation. Second, Approximation 2 converges much faster than Approximation 1. In a multifactor setting the Approximation 2 is preferable since the number of expansion terms greatly affects the speed of computations.
3.3 Comparison with existing methods

In this section we provide several numerical examples to assess the accuracy of both approximations. First, we consider two models where either volatility or interest rates are stochastic. For each model we select the most recent paper that advocates an alternative computational method. For different model parameters we provide three American put prices: the one computed under our analytical approach, the one computed under the alternative approach, and the one computed under the Monte-Carlo approach of Longstaff and Schwartz (2001). The third price is considered as the "true" one since we use a large number of simulated paths. We conclude the section by reporting our results for the model with both stochastic volatility and stochastic interest rates. No reference paper exists for this general case.

3.3.1 Stochastic volatility

The literature on the pricing of American options under the two-factor model with stochastic volatility is relatively limited. The typical approach relies on solving numerically the PDE satisfied by the American option (see Ikonen and Toivanen (2007) and references therein). We take the same numerical examples based on the following Heston model specification:

\[
\begin{align*}
dS_t &= rS_t + \sqrt{v_t}S_t dW_t^{(1)}, \\
dv_t &= \kappa(\bar{v} - v_t)dt + \sigma_v \sqrt{v_t} dW_t^{(2)},
\end{align*}
\]

with \( r = 0.1 \), \( \bar{v} = 0.16 \), \( \sigma_v = 0.9 \), \( \kappa = 5 \), \( \rho_{12} = 0.1 \) and two levels of the spot volatility parameter \( \sigma = \sqrt{v} = 0.25 \) and 0.5.

The model parameters used in the literature are not very "typical". For example, the empirical estimates of the volatility of volatility, and the rate of volatility mean reversion are usually much lower in magnitude (see e.g. Bakshi, Cao and Chen (2000)). Since the size of model parameters might affect the accuracy of the asymptotic expansion we consider two sets of parameters. In Table 3, the second and the forth columns correspond to the model parameters adopted in Ito and Toivanen (2006), and others. The third and the fifth columns correspond to the volatility of volatility parameter and the rate of mean reversion being halved.

Table 3 reports the American put prices computed using the Longstaff and Schwartz (2001) approach based on Monte-Carlo simulations with 200,000 sample paths (100,000 plus 100,000 antithetic), 500 time steps and 50 exercise dates, the results from Ito and Toivanen (2006), and others.

\footnote{Beside regression-based methods, other Monte-Carlo based approaches are available to price American options such as stochastic mesh methods (Broadie and Glasserman (1997)), and duality methods (Haugh and Kogan (2004), Rogers (2002)), see the book of Glasserman (2003) for further details.}

\footnote{We thank Jari Toivanen for providing us with the results of Ito and Toivanen (2006) pricing method for other values of model parameters.}
our approximations. The European put prices for the same model parameters are computed using a closed form formula.

Approximation 2 based on a 5th order expansion yields accurate values for all combinations of model parameters. Approximation 1 performs less impressively for the first set of model parameters with large volatility of volatility and rate of mean reversion. On the other hand, the expansion yields reasonably accurate approximation for more realistic values. A Matlab code takes less than 0.1 seconds to compute an option price using a 5th order expansion with the search algorithm described in Section 2.5.1. The computational time required by the Ito and Toivanen (2006) algorithm for the same magnitude of errors is at least twice as much.

3.3.2 Stochastic interest rates

A separate line of literature deals with pricing American options under stochastic interest rates. The simple and flexible Hull and White (1990) specification is widely used to capture non-flat stochastic term structure. The dynamics of the short-term interest rate is

$$dr_t = (\eta(t) - \gamma r_t) dt + \beta dW_t^{(3)},$$

where \(\eta(t)\) is some deterministic function that is calibrated from the current term structure (see Hull and White (1990)). In applying our method we use the expansion \(\eta = \eta(T) - \eta'(T)\tau + O(\tau^2)\), that allows to write expansion (22) with the following coefficients: \(\alpha_0(r) = \eta(T) - \gamma r\), and \(\alpha_1(r) = -\eta'(T)\).

To check the accuracy of our approximations we use the model parameter values of Menkveld and Vorst (2001). They consider two types of non-flat interest rate term structure: upward sloping and downward sloping. They also vary the correlation between the underlying and the spot interest rate. We use the same model parameters \(\beta = 0.01, \gamma = 0.1\) for the comparison. We consider only the case of zero correlation since the put prices appear not to be very sensitive to this parameter for the chosen small value of the volatility of interest rate parameter \(\beta\). Table 4 shows the at-the-money put option prices for different combinations of the spot volatility and shapes of the term structure. As in the previous section we provide the results based on Longstaø and Schwartz (2001) algorithm with 200,000 sample paths, 500 time steps, and 50 exercise dates as true option prices. Our approximations are again computed using expansions up to the 5th order with the search algorithm described in Section 2.5.1. Observe that both approximations are sufficiently accurate and in most cases are closer to the Monte-Carlo results than the values obtained by Menkveld and Vorst (2001).

3.3.3 Stochastic volatility and stochastic interest rates

In this section we assess the accuracy of our method in the case of a model with stochastic volatility and stochastic interest rates. Since there is no reference paper for that case, we compare our approximations only with the outcome of the Monte-Carlo based approach. We assume an affine specification, and borrow model parameter values from Bakshi, Cao and Chen (2000). The risk-
neutral dynamics is:

\[ dS_t = r_t S_t dt - \sqrt{v_t} S_t dW_t^{(1)}, \]
\[ dv_t = 1.58(0.03 - v_t)dt + 0.20\sqrt{v_t} dW_t^{(2)}, \]
\[ dr_t = 0.26(0.04 - r_t)dt + 0.08\sqrt{r_t} dW_t^{(3)}, \]
\[ \rho_{12} = -0.26, \rho_{13} = \rho_{23} = 0. \]

Table 5 reports the European and American put values for different combination of the spot volatility and time-to-maturity. The American put price is computed using Longstaff and Schwartz (2001) approach with 200,000 sample paths, 500 time steps, and 50 exercise dates. The European put price is computed using a closed form expression. Our approximations are computed using expansions up to the 5th order with the search algorithm described in Section 2.5.1.

Our method yields reasonably accurate option prices. Approximation 2 seems to be more accurate than Approximation 1. To give an idea of the computational advantage of our method, the Matlab code with Longstaff and Schwartz (2001) algorithm takes about 15 minutes to compute one option price. Our approximation takes only 0.1 seconds.

3.4 Effect of stochastic volatility and stochastic interest rates

In this section we take the advantage of our fast pricing algorithm to study the effects of stochastic volatility and stochastic interest rates on the American put price. In particular, we are interested in their effect on the early exercise premium. A simple alternative approximation approach to pricing American options in a multifactor setting is to compute the European option price using known closed form solution while adding the early exercise premium evaluated under the Black-Scholes model\(^6\). In this section we show that such an approach results in an economically significant underpricing. Apart from illustrating the advantage of our approach for option pricing, this study provides new insights on the key determinants of the American option price.

For numerical analysis we assume an affine model specification:

\[ dS_t = r_t S_t dt - \sqrt{v_t} S_t dW_t^{(1)}, \]
\[ dv_t = 0.2\sqrt{v_t} dW_t^{(2)}, \]
\[ dr_t = 0.1\sqrt{r_t} dW_t^{(3)}, \]

with \( r_0 = 0.05, \sigma_0 = 0.2 \). The choice of model parameters is consistent with empirical findings in the literature on stock options (see e.g. Bakshi, Cao and Chen (2000)). The drifts in dynamics of the variance and the spot interest rate are chosen to be zero. This allows to focus on the pure

\(^6\)We thank Liuren Wu for pointing out this approach.
effect of stochastic factors. In a model with only one stochastic factor, the deterministic drifts in
the variance and the spot interest rate can be easily accommodated in a binomial tree.

Figure 5 illustrates the effect of stochastic factors on the early exercise premium. Specifically,
we compute the early exercise premium in model (29), subtract the early exercise premium in the
Black-Scholes model, and divide the result by the option price under model (29). The four graphs
are constructed for four different combinations of correlation parameters $\rho_{12}$ and $\rho_{13}$, taking values
0 or $-0.5$. Note that strong negative correlation between stock price and its volatility is a well-
established fact in the literature, and is known as the leverage effect. The negative correlation
between interest rate and the stock price is theoretically predictable given that the stock price is a
discounted expectation of the future dividend stream.

Observe that when there is no correlation between the underlying and the stochastic factors,
the early exercise premium is not much affected by factor volatilities. The relative error does
not exceed 1% even for options with 1 year to maturity (Figure 5a). If only one of two correlation
parameters $\rho_{12}$ and $\rho_{13}$ is not equal to zero, then the relative error amounts to 3% for 1 year options
(Figure 5b and 5c). If both correlations are negative then it goes further up to 5% (Figure 5d).
These observations suggest the importance of properly taking into account factor volatilities. The
effect of correlation on the early exercise premium is similar to its effect on the implied volatility
skew. Recall that if the stochastic volatility is uncorrelated with the price then there is no implied
volatility skew at-the-money.

Let us also point out some interesting observations. First, the effect of stochastic factors seems
to be more pronounced for long term in-the-money put options. There the magnitude of the early
exercise premium is the largest. Second, the early exercise premium may sometimes decrease after
introducing stochastic factors (Figure 5a and 5c). However, if both correlations are strongly negative
then the factor randomness unambiguously results in increasing early exercise premium.

4 Summary and conclusions

In this paper we describe a new approach to pricing American options under a general setting
with stochastic volatility and stochastic interest rates. This approach is based on an analytical
approximation, which gives it a number of advantages. First, the computational time is small
even under a sophisticated three-factor model. Second, although the approximation is based on
the short-maturity expansion, it appears to be accurate for time-to-maturity up to several years
under typical model parameter values. Third, for hedging and risk management purposes the
computation of option Greeks can be done analytically by computing explicitly derivatives of the
analytical option price formula.
References


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APPENDIX A. Proof of Proposition 1.

Substituting (14) into (11) we arrive at:

\[-nP_n + \theta P_{n\theta} + P_{n\theta \theta} + \sigma P_{n-1\theta} - 2rP_{n-2\theta} = 0, \quad n = 1, 2, \ldots\]  \tag{30}

with \( P_0 = P_{-1} = 0 \). The homogeneous solutions of equation (30) form a two dimensional space. One dimension is spanned by a polynomial solution which does not satisfy the boundary condition (8) at infinity. The other independent solution has the form:

\[ P_n^0(\theta) = p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta). \]  \tag{31}

Let us substitute (31) in the homogeneous part of (30). After some rearrangements we find:

\[
\left( \frac{d^2p_n^0}{d\theta^2} + \theta \frac{dp_n^0}{d\theta} - np_n^0 \right) \Phi(\theta) + \left( -(n + 1)q_n^0 - \theta \frac{dq_n^0}{d\theta} + \frac{d^2q_n^0}{d\theta^2} + 2 \frac{dp_n^0}{d\theta} \right) \phi(\theta) = 0.
\]

It is easy to verify that PDE \( \frac{d^2p_n^0}{d\theta^2} + \theta \frac{dp_n^0}{d\theta} - np_n^0 = 0 \), has polynomial solution

\[ p_n^0(\theta) = \pi_{n0}^0\theta^n + \pi_{n1}^0\theta^{n-2} + \pi_{n2}^0\theta^{n-4} + \ldots \]

with \( \pi_{n0}^0 = 1 \), \( \pi_{n+1}^0 = \frac{(n-2i)(n-2i-1)}{2i+2} \pi_{ni}^0 \). The polynomial solution to \( -(n + 1)q_n^0 - \theta \frac{dq_n^0}{d\theta} + \frac{d^2q_n^0}{d\theta^2} + 2 \frac{dp_n^0}{d\theta} = 0 \), has the form

\[ q_n^0(\theta) = \chi_{n0}^0\theta^{n-1} + \chi_{n1}^0\theta^{n-3} + \chi_{n2}^0\theta^{n-5} + \ldots \]

with

\[ \chi_{n+1}^0 = \frac{\chi_{n0}^0(n-1-2i)(n-2-2i) + 2\pi_{n+1}^0(n-2i-2)}{2n-2i-2}. \]

Let us now find a particular solution \( P_n^1(\theta) \) of (30), which satisfies the boundary condition at infinity. Any solution of (30) with appropriate behavior at the boundary will be given by:

\[ P_n(\theta) = C_n P_n^0(\theta) + P_n^1(\theta), \]

where \( C_n \) is some constant. Let us look for a particular solution \( P_n^1(\theta) \) in the form

\[ P_n^1(\theta) = p_n^1(\theta)\Phi(\theta) + q_n^1(\theta)\phi(\theta). \]

This implies that the general solution is:

\[ P_n(\theta) = C_n \left[ p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta) \right] + p_n^1(\theta)\Phi(\theta) + q_n^1\phi(\theta). \]  \tag{32}

Let us guess that polynomials \( p_n^1 \) and \( q_n^1 \) are as follows:

\[ p_n^1(\theta) = \pi_{n0}^1\theta^n + \pi_{n1}^1\theta^{n-2} + \pi_{n2}^1\theta^{n-4} + \ldots, \]

\[ q_n^1(\theta) = \chi_{n0}^1\theta^{n-1} + \chi_{n1}^1\theta^{n-3} + \chi_{n2}^1\theta^{n-5} + \ldots. \]
After substituting $P_{n-1}$ and $P_{n-2}$ in the form (32) into equation (30) for $P_n^1$ we will obtain a system of two equations:

\[
\frac{d^2p_n^1}{d\theta^2} + \theta \frac{dp_n^1}{d\theta} - np_n^1 + \left( \sigma C_n - 1 \frac{dp_{n-1}^0}{d\theta} + \sigma p_{n-1}^1 - 2rC_n p_{n-2}^0 - 2rp_{n-2}^1 \right) = 0,
\]

\[-(n+1)q_n^0 - \theta \frac{dq_n^0}{d\theta} + \frac{d^2q_n^0}{d\theta^2} + 2 \frac{dp_n^1}{d\theta} + \left( \sigma C_n p_{n-1}^0 + \sigma C_n - 1 \frac{dq_{n-1}^0}{d\theta} - \theta q_{n-1}^1 - 2rC_n q_{n-2}^0 - 2rC_n q_{n-2}^1 \right) = 0.
\]

These equations can be solved as before. In particular we may assume $\pi_{n0}^1 = \pi_{n0}^1 = 0$ since we can safely subtract a homogeneous solution. Here we do not write down the lengthy recursive relationship. In practice the PDE for $P_n^1$ can be solved directly by the substitution of its guessed form.

**APPENDIX B. 4th order expansion of the solution to the modified problem under the Black-Scholes model.**

The solution to the modified problem has the 4th order short-maturity expansion:

\[
P(\theta, \tau; \theta_0) = \sum_{n=1}^{4} \tau \frac{2^2}{C_n} \left\{ C_n \left[ p_n^0(\theta)\Phi(\theta) + q_n^0(\theta)\phi(\theta) \right] + p_n^1(\theta)\Phi(\theta) + q_n^1(\theta)\phi(\theta) \right\},
\]

where

\[
p_0^0(\theta) = \theta, \quad p_1^0(\theta) = 0, \quad q_0^0(\theta) = 1, \quad q_1^0(\theta) = 0,
\]

\[
p_2^0(\theta) = \theta^2 + 1, \quad p_2^1(\theta) = \frac{1}{2\sigma} C_1 (\sigma^2 - 2\mu), \quad q_2^0(\theta) = \theta, \quad q_2^1(\theta) = 0,
\]

\[
p_3^0(\theta) = \theta^3 + 3\theta, \quad p_3^1(\theta) = \frac{1}{\sigma} \left[ C_2 \sigma^2 - 2C_2\mu - rC_1 \sigma \right] \theta, \quad q_3^0(\theta) = \theta^2 + 2,
\]

\[
q_3^1(\theta) = \frac{1}{8\sigma^4} \left\{ 8C_2 \sigma^3 - 16C_2\sigma \mu - 8rC_1 \sigma^2 - 4C_1 \sigma^2 \mu + C_1 \sigma^4 + 4C_1 \mu^2 \right\},
\]

\[
p_4^0(\theta) = \theta^4 + 6\theta^2 + 3,
\]

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\[ p_1^1(\theta) = \frac{1}{2\sigma} \left[ 3C_3\sigma^2 - 6C_3\mu - 2r\sigma C_2 \right] \theta^2 + \frac{1}{4\sigma^2} \left[ 3C_3\sigma^3 + \sigma^4C_2 - 6C_3\sigma \mu - \sigma (-3C_3\sigma^2 + 6C_3\mu + 2r\sigma C_2) + 4C_2\mu^2 - 2\sigma^3rC_1 - 4\sigma^2C_2\mu - 2r\sigma^2C_2 + 4\sigma rC_1\mu \right], \]

\[ q_1^0(\theta) = \theta^3 + 5\theta, \]

\[ q_1^1(\theta) = \frac{1}{48\sigma^3} \left[ 8C_1\mu^3 + 72C_3\sigma^4 - C_1\sigma^6 - 48r\sigma^3C_2 + 6C_1\sigma^4\mu - 12C_1\sigma^2\mu^2 - 144C_3\sigma^2\mu \right] \theta, \]

and

\[ C_1 = (K \theta_0 \sigma)^{-1} (\Phi_0 \theta_0 + \phi_0), \]

\[ C_2 = - (\Phi_0 C_1 \sigma^2 - 2\Phi_0 C_1 \mu + K \theta_0^2 \sigma^3) \left[ 2\sigma (\Phi_0 \theta_0^2 + \Phi_0 + \phi_0 \theta_0) \right]^{-1}, \]

\[ C_3 = \left[ 24\sigma^2 \left( \Phi_0 \theta_0^3 + 3 \Phi_0 \theta_0 + \phi_0 \theta_0^2 + 2 \phi_0 \right) \right]^{-1} \times \left( -24 \Phi_0 \theta_0 \sigma^3 C_2 + 48 \Phi_0 \theta_0 \sigma C_2 \mu + 24 \Phi_0 \theta_0 \sigma^2 r C_1 -24 \phi_0 C_2 \sigma^3 + 48 \phi_0 C_2 \sigma \mu + 24 \phi_0 r C_1 \sigma^2 + 12 \phi_0 C_1 \sigma^2 \mu - 3 \phi_0 C_1 \sigma^4 - 12 \phi_0 C_1 \mu^2 + 4 K \theta_0^3 \sigma^5 \right), \]

\[ C_4 = \left[ 48\sigma^3 \left( \Phi_0 \theta_0^4 + 6 \Phi_0 \theta_0^2 + 3 \Phi_0 + \phi_0 \theta_0^3 + 5 \phi_0 \theta_0 \right) \right]^{-1} \times \left( 72 \Phi_0 \sigma^4 \theta_0^2 C_3 - 144 \Phi_0 \sigma^2 \theta_0^2 C_3 \mu - 48 \Phi_0 \sigma^3 \theta_0^2 r C_2 + 48 \Phi_0 \sigma C_2 \mu^2 + 12 \Phi_0 \sigma^2 C_2 + 72 \Phi_0 \sigma^3 C_3 - 144 \Phi_0 \sigma^2 C_3 \mu - 48 \Phi_0 \sigma^3 r C_2 - 24 \Phi_0 \sigma^4 r C_1 -48 \Phi_0 \sigma^3 C_2 \mu + 48 \Phi_0 \sigma^2 r C_1 \mu + 8 \phi_0 \theta_0 C_1 \mu^3 + 72 \phi_0 \theta_0 C_3 \sigma^4 - \phi_0 \theta_0 C_1 \sigma^6 -48 \phi_0 \theta_0 r \sigma^3 C_2 - 12 \phi_0 \theta_0 C_1 \mu^2 \sigma^2 - 144 \phi_0 \theta_0 C_3 \sigma^2 \mu + 6 \phi_0 \theta_0 C_1 \sigma^4 \mu + 2 K \theta_0^4 \sigma^7 \right), \]

\[ \mu = r - \delta. \]
Figure 1. The solution to the modified problem
Both graphs plot the solution to the modified problem as a function of the early exercise level of the normalized moneyness \( y \). Option parameters are: \( S = K = 100, \tau = 1 \). Volatility is at \( \sigma = 0.2 \). The solution is denoted by \( P(0,1; y) \) \( (\theta = 0, \tau = 1) \). The American put price is denoted by \( P \) and the European put price is denoted by \( P(0,1; \infty) \). In case (b), the American put price is equal to the European put price, and the maximum of \( P(0,1; y) \) is achieved at \( y = \infty \).
Figure 2. Convergence of the asymptotic expansion in the Black-Scholes model
The graph shows the relative error of the approximation of the American put option price under the Black-Scholes for three levels of moneyness. Model parameters are $r = 0.05$, $\sigma = 0.2$, $\delta = 0$. 
Figure 3. Approximation of the American put option delta at the exercise

The graph shows approximate deltas of the American put option at the early exercise price. The approximation is found by taking an analytical derivative of the approximation of the American put.
Figure 4. Convergence of the approximations under stochastic volatility

Graph (a) shows the convergence of the direct approximation of the American put price (Approximation 1). Graph (b) shows the convergence of the alternative approximation (Approximation 2), where only the early exercise premium is approximated. The variance follows a square-root process:

\[ dv = 2(0.04 - v)dt + 0.2\sqrt{v} dW^{(2)}, \quad \rho_{12} = -0.5. \]
Figure 5. The effect of stochastic volatility and stochastic interest rates on the early exercise premium

The graphs show the impact of the introduction of two additional stochastic factors. They plot the increase in the early exercise premium relative to the option price for different combinations of correlation parameters. The option moneyness $K/S$ runs from 0.8 to 1.2 and time-to-maturity $\tau$ - from 1 month to 1 year. The model specification is:

$$
\begin{align*}
    dS_t &= r_S S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \\
    dv_t &= 0.2 \sqrt{v_t} dW_t^{(2)}, \\
    dr_t &= 0.1 dW_t^{(3)}, \\
    \rho_v dt &= dW^{(4)}_t dW^{(1)}_t.
\end{align*}
$$

(a) $\rho_{12} = \rho_{13} = 0$

(b) $\rho_{12} = -0.5, \rho_{13} = 0$

(c) $\rho_{12} = 0, \rho_{13} = -0.5$

(d) $\rho_{12} = -0.5, \rho_{13} = -0.5$
Table 1. American call option prices and their approximations under the Black-Scholes model.
The table compares option price bounds of Broadie and Detemple (1996) with our approximation based on asymptotic expansions of different orders. Lower and Upper bounds refer to option price bounds reported in Tables 1 and 2 of Broadie and Detemple (1996). “True value” is the option price computed with a 15’000-step binomial tree. Here all options have strike price $K = 100$. Time-to-maturity $\tau$, asset price $S$, interest rate $r$, volatility $\sigma$, and dividend rate $\delta$ are indicated in the table.

<table>
<thead>
<tr>
<th>Option parameters</th>
<th>$S$</th>
<th>$r = 0.03$, $\sigma = 0.2$, $\delta = 0.07$</th>
<th>$r = 0.03$, $\sigma = 0.4$, $\delta = 0.07$</th>
<th>$r = 0$, $\sigma = 0.3$, $\delta = 0.07$</th>
<th>$r = 0.07$, $\sigma = 0.3$, $\delta = 0.03$</th>
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<td></td>
<td>80</td>
<td>90</td>
<td>100</td>
<td>110</td>
<td>120</td>
</tr>
<tr>
<td>European</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tau = 0.5$ years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>True value</td>
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<td>1.386</td>
<td>4.783</td>
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</tr>
<tr>
<td>$\tau = 3$ years</td>
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<td></td>
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</table>
Table 2. Put option prices and their approximations under the Black-Scholes model.
The table compares the approach of Bunch and Johnson (2000) with our approximation based on a 4-the order asymptotic expansion. Bunch & Johnson refers to results reported in Table II of Bunch and Johnson (2000). “True value” is the price computed with a 10’000-step binomial tree. Here asset price $S = 40$ and interest rate $r = 0.0488$. Time-to-maturity $\tau$ and asset price volatility $\sigma$ are indicated in the table.

<table>
<thead>
<tr>
<th>Method</th>
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<th>$\sigma = 0.4$</th>
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<td>$\tau = 1/3$</td>
<td>$\tau = 7/12$</td>
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<td>$K = 35$</td>
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<tr>
<td>European put</td>
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<td>0.196</td>
<td>0.417</td>
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<tr>
<td>Bunch &amp; Johnson</td>
<td>0.006</td>
<td>0.200</td>
<td>0.432</td>
</tr>
<tr>
<td>300-step tree</td>
<td>0.006</td>
<td>0.200</td>
<td>0.433</td>
</tr>
<tr>
<td>True value</td>
<td>0.006</td>
<td>0.200</td>
<td>0.433</td>
</tr>
<tr>
<td>$K = 40$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>European put</td>
<td>0.840</td>
<td>1.522</td>
<td>1.881</td>
</tr>
<tr>
<td>Bunch &amp; Johnson</td>
<td>0.853</td>
<td>1.581</td>
<td>1.992</td>
</tr>
<tr>
<td>300-step tree</td>
<td>0.853</td>
<td>1.581</td>
<td>1.990</td>
</tr>
<tr>
<td>True value</td>
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<td>1.580</td>
<td>1.990</td>
</tr>
<tr>
<td>$K = 45$</td>
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<td></td>
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<tr>
<td>4th order</td>
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<td>5.085</td>
<td>5.261</td>
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<tr>
<td>Bunch &amp; Johnson</td>
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<td>5.091</td>
<td>5.265</td>
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<tr>
<td>300-step tree</td>
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<td>5.088</td>
<td>5.267</td>
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<tr>
<td>True value</td>
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<td>5.088</td>
<td>5.267</td>
</tr>
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</table>
### Table 3. Put option prices under stochastic volatility and their approximations

Option prices are computed for the Heston model:

\[
\begin{align*}
    dS_t &= rS_t dt + \sqrt{v_t}S_t dW^{(1)}_t, \\
    dv_t &= \kappa(\bar{v} - v_t)dt + \sigma_v \sqrt{v_t}dW^{(2)}_t,
\end{align*}
\]

with \( r = 0.1, \kappa = 5 \) (or 2.5), \( \bar{v} = 0.16, \sigma_v = 0.9 \) (or 0.45), \( \rho_{12} = 0.1 \). Time-to-maturity \( \tau \) is 3 months and strike price \( K \) is 10. The model parameters are borrowed from Ito and Toivanen (2006), and the spot variance is denoted by \( \nu \). Monte-Carlo refers to Longstaff and Schwartz (2000) algorithm with 200,000 sample paths, 500 time steps and 50 exercise dates. Our approximation is computed with a 5-th order asymptotic expansion. Ito&Toivanen corresponds to results presented in Tables 1 and 2 of Ito and Toivanen (2006). European put price is computed using a closed form formula.

<table>
<thead>
<tr>
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<th>( \nu = 0.0625 )</th>
<th>( \nu = 0.25 )</th>
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<td></td>
<td>( \sigma_v = 0.9 )</td>
<td>( \sigma_v = 0.45 )</td>
</tr>
<tr>
<td></td>
<td>( \kappa = 5 )</td>
<td>( \kappa = 2.5 )</td>
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<tr>
<td>( S = 9 )</td>
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<td></td>
</tr>
<tr>
<td>Approximation 1 (5-th order)</td>
<td>1.081</td>
<td>1.072</td>
</tr>
<tr>
<td>Approximation 2 (5-th order)</td>
<td>1.111</td>
<td>1.077</td>
</tr>
<tr>
<td>Ito&amp;Toivanen</td>
<td>1.108</td>
<td>1.077</td>
</tr>
<tr>
<td>Monte-Carlo</td>
<td>1.107</td>
<td>1.075</td>
</tr>
<tr>
<td>European put</td>
<td>1.048</td>
<td>1.009</td>
</tr>
<tr>
<td>( S = 10 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Approximation 1 (5-th order)</td>
<td>0.534</td>
<td>0.475</td>
</tr>
<tr>
<td>Approximation 2 (5-th order)</td>
<td>0.521</td>
<td>0.478</td>
</tr>
<tr>
<td>Ito&amp;Toivanen</td>
<td>0.520</td>
<td>0.479</td>
</tr>
<tr>
<td>Monte-Carlo</td>
<td>0.521</td>
<td>0.478</td>
</tr>
<tr>
<td>European put</td>
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<td>0.459</td>
</tr>
<tr>
<td>( S = 11 )</td>
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<td></td>
</tr>
<tr>
<td>Approximation 1 (5-th order)</td>
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<td>0.174</td>
</tr>
<tr>
<td>Approximation 2 (5-th order)</td>
<td>0.214</td>
<td>0.178</td>
</tr>
<tr>
<td>Ito&amp;Toivanen</td>
<td>0.214</td>
<td>0.178</td>
</tr>
<tr>
<td>Monte-Carlo</td>
<td>0.214</td>
<td>0.177</td>
</tr>
<tr>
<td>European put</td>
<td>0.208</td>
<td>0.173</td>
</tr>
</tbody>
</table>
Table 4. Put option prices under stochastic interest rates and their approximations

The put option prices are computed for the constant volatility model with interest rates following the Hull and White model:

\[ dr_t = (\eta(t) - \gamma r_t) dt + \beta dW_t^{(s)}. \]

The model parameters are \( \gamma = 0.1, \beta = 0.01 \). Time-to-maturity \( \tau \) is one year, stock price and strike price \( S = K = 100 \), the stock price volatility is denoted by \( \sigma \). USTS refers to an upward sloping term structure, and DSTS refers to the downward sloping term structure. The model parameters and term structure specifications are borrowed from Menkveld and Vorst (2001). Monte-Carlo refers to the Longstaff and Schartz (2000) algorithm with 200'000 sample paths, 500 time steps, and 50 exercise dates. Our approximation is computed with a 5-th order asymptotic expansion. Menkveld&Vorst refers to the results reported in Table 5 of Menkveld and Vorst (2001). European put prices are also taken from the same table.

<table>
<thead>
<tr>
<th>Method</th>
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<th></th>
<th>DSTS</th>
<th></th>
<th></th>
</tr>
</thead>
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<tr>
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<td>( \sigma = 0.3 )</td>
<td>( \sigma = 0.2 )</td>
<td>( \sigma = 0.5 )</td>
<td>( \sigma = 0.3 )</td>
<td>( \sigma = 0.2 )</td>
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<td>18.00</td>
<td>10.36</td>
<td>6.52</td>
<td>17.84</td>
<td>10.20</td>
<td>6.37</td>
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<tr>
<td>Approximation 2 (5th order)</td>
<td>17.98</td>
<td>10.35</td>
<td>6.52</td>
<td>17.86</td>
<td>10.22</td>
<td>6.39</td>
</tr>
<tr>
<td>Merkved&amp;Vorst</td>
<td>17.93</td>
<td>10.33</td>
<td>6.53</td>
<td>17.82</td>
<td>10.20</td>
<td>6.39</td>
</tr>
<tr>
<td>Monte-Carlo</td>
<td>17.98</td>
<td>10.36</td>
<td>6.49</td>
<td>17.88</td>
<td>10.25</td>
<td>6.41</td>
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<td>17.55</td>
<td>9.92</td>
<td>6.09</td>
<td>17.55</td>
<td>9.92</td>
<td>6.09</td>
</tr>
</tbody>
</table>
Table 5. Put option prices under stochastic volatility and stochastic interest rates and their approximations

Option prices are computed for an affine model with stochastic volatility and stochastic interest rates:

\[
dS_t = r_s dt + \sqrt{v_t} S_t dW^{(1)}_t, \\
dv_t = 1.58(0.03 - v_t)dt + 0.2\sqrt{v_t} dW^{(2)}_t, \\
dr_t = 0.26(0.04 - r_t) + 0.08\sqrt{r_t} dW^{(3)}_t,
\]

with \( \rho_{12} = -0.26, \rho_{13} = \rho_{23} = 0 \). The stock price is \( S = 100 \) and spot interest rate is \( r = 0.04 \). Time-to-maturity is denoted by \( \tau \), and the spot volatility is denoted by \( \sigma = \sqrt{v} \). Monte-Carlo refers to the Longstaff and Schwarz (2000) algorithm with 200,000 sample paths, 500 time steps, and 50 exercise dates. Our approximation is computed with a 5-th order asymptotic expansion.

<table>
<thead>
<tr>
<th>Method</th>
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<th>( \sigma = 0.3 )</th>
<th>( \sigma = 0.4 )</th>
</tr>
</thead>
<tbody>
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<td>( \tau = 1/12 )</td>
<td>( \tau = 1/4 )</td>
<td>( \tau = 1/2 )</td>
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<td>Approximation 1 (5th order)</td>
<td>0.076</td>
<td>0.403</td>
<td>0.970</td>
</tr>
<tr>
<td></td>
<td>0.613</td>
<td>1.622</td>
<td>2.854</td>
</tr>
<tr>
<td></td>
<td>1.381</td>
<td>2.822</td>
<td>4.610</td>
</tr>
<tr>
<td>Approximation 2 (5th order)</td>
<td>0.076</td>
<td>0.403</td>
<td>0.970</td>
</tr>
<tr>
<td></td>
<td>0.612</td>
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<td>2.904</td>
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<td>1.357</td>
<td>2.819</td>
<td>4.573</td>
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<tr>
<td>American put (Monte-Carlo)</td>
<td>0.075</td>
<td>0.404</td>
<td>0.969</td>
</tr>
<tr>
<td></td>
<td>0.621</td>
<td>1.627</td>
<td>2.903</td>
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<td>1.351</td>
<td>2.819</td>
<td>4.573</td>
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<td></td>
<td>0.604</td>
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<td>3.228</td>
<td>4.329</td>
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