Spectral Iterative Estimation of Tempered Stable Stochastic Volatility Models and Option Pricing

CARLO FAVERO
IGIER and Bocconi University

JUNYE LI
PhD Program in Economics
Bocconi University

FULVIO ORTU
Institute of Quantitative Methods
Bocconi University

ADDRESS FOR CORRESPONDENCE:
Junye Li
PhD Program in Economics
Bocconi University
Via Sarfatti 25, 20136
Milan, Italy
Tel: 0039 02 5836 5932
Email: junye.li@phd.unibocconi.it
Abstract

This paper addresses alternative option pricing models and their estimation. The stock price dynamics is modeled by taking into account both stochastic volatility and jumps. Jumps are mimicked by the tempered stable process and stochastic volatility is introduced by time changing the stochastic process. We propose a characteristic function based iterative estimation method, which overcomes the problem of nontratable probability density functions of our models and eases computational difficulty related to other methods. The estimation results and option pricing performance indicate that the infinite activity stochastic volatility model is more preferable than the finite activity model. We also make an extension to investigate double-jump model by introducing jumps in the variance rate process.

Keywords: Tempered Stable Process, Stochastic Volatility, Time Change, Option Pricing, Iterative Method, Joint CCF-CGMM.

JEL Classification: C13, C22, G10, G12, G13

1 Introduction

Stochastic volatility and jumps in stock price process are well documented. On the one hand, they are inherent components of stock price dynamics (Bollerslev et al., 1994; Merton, 1976); on the other hand, jumps and stochastic volatility play important roles in explaining distributional characteristics of returns and implied volatility smile/skew of options. Bakshi et al.(1997) and Bates (2000) find that even though stochastic volatility alone could explain distributional skewness and leptokurtosis of stock returns to a certain extent, its ability to price short-maturity options is limited. By introducing jumps in stock price modeling, this limitation is largely overcome. Jumps mainly affect short-maturity options, while stochastic volatility mainly affects long-maturity options.

Much work has been done on jump-diffusion stochastic volatility models (Bakshi et al., 1997; Bates, 1996, 2000; Pan, 2002; Andersen et al., 2002). These models regard jumps as rare events and use compound poisson process to mimic jumps. The stochastic volatility is modeled by a mean-reverting square-root process (Heston, 1993). Even though these jump-diffusion stochastic volatility models work well relatively in fitting stock price process and pricing options, there is an undesirable assumption. Jumps actually are not rare events. By observing time-series evolution of stock prices, we find that the stock price process is accompanied not only by large rare jumps, but also by a lot of small jumps. Based on this observation, another alternative
models are being developed. These models use infinite activity Lévy processes to capture large jumps as well as small jumps in stock price dynamics. Stochastic volatility is usually introduced by time-changing these Lévy processes (Carr et al., 2003; Carr and Wu, 2004).

In this paper, we introduce the tempered stable process, which is a Lévy process having six parameters at most in its Lévy density controlling not only the “location” and “scale”, but also “skewness” and “kurtosis” of its distribution. Depending on different values of the stable index of the Lévy density, the tempered stable process can be an infinite activity process, which generates infinite number of (large and small) jumps or a finite activity process, which generates only finite number of large jumps. The tempered stable process can also exhibit infinite or finite variation with different values of the stable index. It, in fact, includes as special cases many other stochastic processes, such as the compound poisson process and the variance gamma process. Therefore, with this process we could unify jump-diffusion model and infinite activity model into one framework but with rich structure. In the financial application, the tempered stable process is introduced by Carr et al (2002) under the name of CGMY process and further studied by Wu (2006) and Li (2007).

We apply the time-change approach to introduce stochastic volatility (Carr et al., 2003). Given a strictly positive right continuous with left limit stochastic process \( \nu_t \), we define a stopping time as

\[
T_t = \int_0^t \nu_s - ds,
\]

which is finite almost surely. Intuitively, we could think of \( t \) as calendar time and \( T_t \) as business time. The variable \( \nu_t \) reflects the intensity of economic activity and we call it variance rate. Stochastic volatility is generated by replacing calendar time \( t \) with business time \( T_t \). For a stochastic process \( X(t) \), its time-changed counterpart is defined by

\[ X_{T_t} = X(T_t). \]

When we use this time-changed stochastic process to model the stock price process, we need to study its distributional characteristics. If we assume independence between \( \nu_t \) and \( X(t) \), through iterated expectation we could obtain the conditional characteristic function of \( X_{T_t} \)

\[
\phi_X(u, \tau) = E[e^{iuX_{T_t}} | \mathcal{F}_t] = E[ e^{-c T_t} | \mathcal{F}_t] \\
= E[ e^{-c \int_0^{T_t} \nu_s ds} | \mathcal{F}_t] \quad (1)
\]
and the joint conditional characteristic function of $X_{T+\tau}$ and $\nu_{t+\tau}$

$$\phi_{X,\nu}(u_1, u_2, \tau) = E[e^{iu_1X_{T+\tau} + iu_2\nu_{t+\tau}} | \mathcal{F}_t] = E[e^{-cT} e^{iu_2\nu_{t+\tau}} | \mathcal{F}_t]$$

$$= E[e^{-c \int_{t}^{t+\tau} \nu_s ds} e^{iu_2\nu_{t+\tau}} | \mathcal{F}_t],$$

(2)

where $c$ is related to the characteristic function of $X(t)$. Specifically, in this paper when computing option prices, we need the conditional characteristic function under the risk-neutral measure and when implementing estimation, we need the joint conditional characteristic function under the objective measure. Note that the calculation of (1) and (2) is equivalent to finding the transforms in the sense of Duffie, Pan and Singleton (2000) if $\nu_t$ falls into the class of affine jump-diffusion processes.

We firstly build a general model with the exponential time-changed Brownian motion and tempered stable process. Time-changing is conducted with the square-root process. The variance rate is allowed to be correlated with the return process. We have mentioned that the tempered stable process takes on different properties with respect to the stable index. We then investigate these properties by imposing different restrictions on the parameters. In particular, we want to see whether the tempered stable process acts as an infinite activity process or a finite activity process when both jump and stochastic volatility are taken into account in modeling stock price dynamics. We also want to empirically compare the performance of jump-diffusion stochastic volatility model and infinite activity stochastic volatility model in pricing options. Furthermore, we make an extension of our general model by introducing jump component in the variance rate process and investigate Double-Jump model.

A major difficulty in continuous-time financial modeling is the lack of efficient tools for estimating and making inference with discretely observed samples, especially when models have latent factors and jumps. This is particularly striking for the models studied in this paper. The frequently used simulation-based methods are difficult to implement since the models are hard to simulate. The traditional GMM is computationally demanding since we have to calculate high order derivatives. Fortunately, for the models we study in this paper the analytical characteristic functions are always obtainable. The characteristic function is equivalent to the probability density function and they contain the same information. We could thus consider to directly use the characteristic function to implement estimation. Since our models contain the latent factors, we propose a characteristic function based iterative method to jointly estimate the models.
through taking advantage of information contained both in stock market and options markets. With an initial parameter guess, we firstly back out unobserved variance rates from options and regard them as if they are observable, and then implement characteristic function based GMM with a continuum of moment conditions (Carrasco et al., 2007). With estimates obtained in the previous step, we repeat this fashion many times until a certain criterion is reached. With this iterative method, we not only obtain consistent estimates of model parameters, but also identify the market prices of risks as well as filter out a sequence of state variables which is the best proxy for the true ones.

The estimation using characteristic function is not novel. It is investigated as early as in 1970’s (Feuerverger and Mureika, 1977) and further discussed by Feuerverger and McDunnough (1981a, 1981b) and Feuerverger (1990). The method is redeveloped recently for the estimation of continuous-time financial models by Singleton (2001), Jiang and Knight (2002) and Chacko and Viceira (2003). Carrasco et al. (2007) extend the method by using a continuum of moment conditions and prove improvement in efficiency upon discrete moment conditions. These approaches are very useful for estimating models not containing unobserved state variables like stochastic volatility. We overcome this problem through joint use of stock prices and options. The investigation of joint estimation with time series data on stock prices and panel data on options is one of the frontiers in empirical work. Renault and Touzi (1996), Pastorello et al. (2000), and Chernov and Ghysels (2000) simply proxy unobserved volatility with Black-Scholes implied volatility to estimate stochastic volatility models jointly. When jumps are introduced to the stock price process, however, this approach becomes inappropriate. Garcia et al. (2006) jointly estimate the stochastic volatility models using both option prices and high-frequency underlying prices based on series expansions of option prices and implied volatilities and on the method of moments which takes advantage of tractable moments of realized volatility. This method suffers from the same problem as previous ones. Pan (2002) advocates an implied-state GMM to focus directly on joint dynamics of stock return and near-the-money short maturity options. Pastorello et al. (2003) proposes a general iterative and recursive method on estimating structural nonadaptive models. This method actually encompasses a large set of implied state methodologies including the implied-state GMM. Our method is similar to those of Pan (2002) and Pastorello et al. (2003) except that we directly use characteristic function. The direct use of characteristic function in estimation is not trivial. It makes the estimation of many models like those studied in this paper feasible and avoids the demanding tasks of simulation of the models.
and computation of high-order derivatives.

The remaining of the paper is organized as follows. Section 2 gives a detailed derivation of the models. We derive the conditional characteristic function of return under the risk-neutral measure for option pricing and the joint conditional characteristic function of return and variance rate under the objective measure for estimation; section 3 describes the characteristic function based iterative joint estimation method used in this paper. A simple Monte Carlo study is conducted with Heston’s model; section 4 presents the data which include both stock prices and options; section 5 discusses the estimation results and evaluates the models; section 6 extends the model by introducing jump component in the variance rate process; and finally section 7 concludes the paper. Proofs of the propositions and algorithms are given in appendix.

2 Models

In this section, we introduce our general model which is built on a time-changed Brownian motion and tempered stable process. By doing so, both stochastic volatility and jumps are incorporated in the stock price dynamics. Since we eventually aim at option pricing, we firstly specify the risk-neutral stock price dynamics in subsection 2.1 and then derive the objective one with the definition of the market price of risk in subsection 2.2. We derive both the risk-neutral conditional characteristic function of return for option pricing and the objective joint conditional characteristic function of return and variance rate for model estimation.

2.1 Risk-Neutral Stock Price Dynamics

Under a given probability space \((\Omega, \mathcal{F}, Q)\) and the complete filtration \((\mathcal{F}_t)_{t \geq 0}\), we introduce the tempered stable process \(X_t\) to model the stock price dynamics. The tempered stable process \(X_t\) is a Lévy process on \(R\) with Lévy density defined as:

\[
v(x) = c \frac{e^{-\lambda_+ x}}{x^{1+\alpha}} \mathbb{1}_{x > 0} + e^{-\lambda_- |x|} \frac{1}{|x|^{1+\alpha}} \mathbb{1}_{x < 0}
\]

where \(c > 0\) and \(\lambda_+, \lambda_- > 0\). To guarantee the finite quadratic variation, the stable index \(\alpha\) should be less than 2. The Lévy density \(v(x)\) measures arrival rate of jumps with size \(x\) defined
on $\mathbb{R}^0$ (real line without zero). Its characteristic function has the form of, with $\alpha \neq 1$ and $\alpha \neq 0$,

$$
\phi_X(u) = E[e^{iuX(t)}] = e^{-t\psi(u)},
$$

$$
\psi_X(u) = -c\Gamma(-\alpha)\left[(\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha\right],
$$

where $\psi(u)$ is the characteristic exponent, $u \subseteq \mathbb{R}$, $\mathbb{R}$ is the real line and $\Gamma(\cdot)$ is a gamma function.

The expectation is assumed to be well-defined in $D$. When $\alpha = 1$ or $\alpha = 0$, the process has a different form of characteristic function. Specifically, the characteristic exponent is

$$
\psi_X(u) = c[iu/\lambda_+ + \log(1 - iu/\lambda_+)] + c[-iu/\lambda_- + \log(1 + iu/\lambda_-)],
$$

when $\alpha = 0$ and it becomes

$$
\psi_X(u) = -c(\lambda_+ - iu)\log(1 - iu/\lambda_+) - c(\lambda_- + iu)\log(1 + iu/\lambda_-).
$$

when $\alpha = 1$.  

The parameters in the Lévy density of the tempered stable process play different roles in controlling its distribution. $c$ measures the overall and relative frequency of jumps; $\lambda_+$ and $\lambda_-$ determine the tails behavior of the Lévy measure. They govern how fast the tails decay and lead to skewed distribution when they are not the same; and the stable index $\alpha$ governs how the process evolves between big jumps. Specifically, if $\alpha < 0$, the tempered stable process becomes a finite activity process, while if $\alpha \geq 0$, it is an infinite activity process; when $\alpha < 1$, the tempered stable process exhibits finite variation, whereas when $1 \leq \alpha \leq 2$, it has infinite variation.

The stock price process under the risk-neutral measure is modeled by an exponential time-changed Brownian motion and tempered stable process:

$$
S_t = S_0 \exp\left\{(r - q)t + \left[W_{T_t} - k_W(1)T_t\right] + \left[X_{T_t} - k_X(1)T_t\right]\right\},
$$

$$
T_t = \int_0^t \varphi_s\,ds,
$$

where $r$ is a constant risk-free rate, $q$ is the dividend yield, $W_t$ is a standard Brownian motion, $X_t$ is the tempered stable process, $T_t$ is the stochastic business time, $\varphi_t$ is the variance rate, and $k_W(1)$ and $k_X(1)$ are convexity adjustments. For any stochastic process $Y_t$, the convexity
adjustment could be derived from its cumulant exponent \( k(s) \), which is defined as

\[
k(s) \equiv \frac{1}{t} \log(E[e^{sY_t}]) \equiv -\psi_Y(-is),
\]

(9)

where \( \psi_Y(\cdot) \) is the characteristic exponent of the process \( Y_t \). Apparently, the convexity adjustment of Brownian motion is \( k_W(1) = \frac{1}{2} \) since it has a normal distribution. The convexity adjustment of the tempered stable process \( k_X(1) \) can be derived from its characteristic exponent (4), (5) or (6).

Time-changing a Lévy process means randomly changing calendar time on which this Lévy process runs. It is a standard technique to generate stochastic volatility (Carr et al., 2003). This randomly changed time could be regarded as business time or trading time. The randomness in business time generates the stochastic volatility. In fact, with the time-changing approach, we introduce not only stochastic volatility, but also stochastic higher moments such as skewness and kurtosis. The function \( t \mapsto T_t \) should be a nonnegative, nondecreasing and right-continuous with left limit process. The nonnegativity and nondecreasing are to imitate the properties of calendar time. To guarantee the nonnegativity and nondecreasing of \( T_t \), the variance rate \( \nu_t \) ought to be a nonnegative process.

A well-known nonnegative process we can use for \( \nu_t \) is that used by Cox, Ingersoll and Ross (CIR process; 1985) for interest rate modeling and later generalized by Heston (1993) for stochastic volatility modeling. Under the risk-neutral measure, this process has the following stochastic differential equation (SDE):

\[
d\nu_t = \kappa(\theta - \nu_t)dt + \sigma \sqrt{\nu_t}dZ_t,
\]

(10)

where if \( \kappa > 0 \), \( \kappa \) is the rate of mean-reversion; \( \theta \) is the long-run mean of the variance rate; and \( \sigma \) is a variation parameter. \( Z_t \) is an another standard Brownian motion. SDE (10) has an analytical solution and the conditional expected value and conditional variance of \( \nu_t \) given \( \nu_s \) are

\[
E[\nu_t|\nu_s] = \nu_s e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}),
\]

(11)

\[
Var[\nu_t|\nu_s] = \frac{\nu_s \sigma^2}{\kappa} (e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}) + \frac{\theta \sigma^2}{2\kappa} (1 - e^{-\kappa(t-s)})^2.
\]

(12)
We note that the conditional variance of $v_t$ is increasing with respect to $\sigma$, whereas it is decreasing with $\kappa$. If $\sigma^2 < 2\kappa \theta$, the process (10) never reaches zero.

Brownian motion $Z_t$ could be allowed to correlate with $W_t$ with instantaneous correlation $[dW_t dZ_t] = \rho dt$, where $\rho \in [-1, 1]$. This is to accommodate the so-called leverage effect of the diffusion part. The leverage effect of jump is actually inherent in time-changed model because in the time of high variance rate, business time flows faster and price jumps occur at a higher rate. The dynamics of $v_t$ captures variation of the overall risk in the market.$^2$

It is possible to time change two processes separately by using different variance rate processes. To keep parsimony of the model, in this paper we use the same variance rate process to time-change both Brownian motion and the tempered stable process. We don’t need to time change the drift term since it is deterministic and is defined by no arbitrage. Under these specifications, we obtain a model of stock price process built on the jump and stochastic volatility.

In the following, we refer to this model as “LTS-SV” model. This general model can flexibly explain negative skewness and leptokurtosis in distribution of stock returns. Negative skewness can arise either from the difference of tail parameters of the tempered stable process or from negative correlation between variance rate and return process; The positive excess kurtosis can arise either from high jump frequency induced by the tempered stable process or from volatile variance rate.

Since the return process is correlated with the variance rate process, we firstly internalize this correlation with the approach proposed by Carr and Wu (2004) and then derive the conditional characteristic function of log returns as described in Section 1.

**PROPOSITION 1:** The conditional characteristic function of log return $R_{t+\tau} = \ln(S_{t+\tau}/S_t)$ in LTS-SV model with variance rate process specified by (10) under the risk-neutral measure is

$$\phi_R(u; \tau, v_t) \equiv E^Q[e^{iuR_{t+\tau}}|\mathcal{G}_t] = e^{iu(r-q)\tau} \frac{A(u, \tau) + (\gamma - \kappa^*)\tau}{B(u, \tau)},$$

where

$$A(u, \tau) = -\frac{\kappa \theta}{\sigma^2} \left[2 \log \left(1 - \frac{(\gamma - \kappa^*)(1 - e^{-\gamma \tau})}{2\gamma}\right) + (\gamma - \kappa^*)\tau\right],$$

$$B(u, \tau) = \frac{2[\varphi_W(u) + \varphi_X(u)](1 - e^{-\gamma \tau})}{(\gamma - \kappa^*)(1 - e^{-\gamma \tau}) - 2\gamma}.$$
\[
\varphi_W(u) = \frac{1}{2}(iu + u^2), \\
\varphi_X(u) = \psi_X(u) + iuk_X(1), \\
\kappa^* = \kappa - iu\rho\sigma, \\
\gamma = \sqrt{(\kappa^*)^2 + 2\sigma^2[\varphi_W(u) + \varphi_X(u)]}. 
\]

PROOF: Proof is given in Appendix A. □

Note that the characteristic function (13) depends on unobserved variance rate \( \nu_t \). We also note that now the information flow is modeled by the complete filtration \((\mathcal{F}_t)_{t \geq 0}\) generated by business time sigma algebra \( \mathcal{F}_{T_t} \). With the above characteristic functions, we could use fast Fourier transform (FFT) to numerically compute option price if we can observe variance rate. Option pricing with FFT is proposed by Carr and Madan (1999). Chourdakis (2005) advocates the fractional Fourier transform (FRFT) in pricing options. It is demonstrated that FRFT is more efficient than FFT in the sense of computational precision by careful selection of integration upper bound and grid sizes of characteristic index and log strike. In this paper, we apply FRFT to option pricing.

2.2 Market Prices of Risks and Objective Joint CCF

In order to specify the objective stock price process, we should define the market prices of risks. By introducing stochastic volatility and jumps to stock price process, the market is no longer complete. This means that there may exist many equivalent martingale measures which can guarantee absence of arbitrage. This feature may produce extra difficulty and complexity in the measure change, since the objective dynamics could be extremely different from that under the risk-neutral measure.

We are interested in structure-preserving change of measure because it preserves tractability and the same structure under both measures. Under the objective measure \( P \), which is assumed to be absolutely continuous with respect to \( Q \), we propose the following definitions of risk premia and stock price and variance rate dynamics,

\[
S_t = S_0 \exp \left\{ (r - q)t + \pi_W T_t + \left[ k_X(1) - k_X(1) \right] T_t \\
+ \left[ W_{P_t}^P - \frac{1}{2} T_t \right] + \left[ X_{P_t}^P - k^P(1) T_t \right] \right\}, \quad (14)
\]
and

\[ dv_t = [\kappa(\theta - v_t) + \pi_{v}v_t]dt + \sigma \sqrt{v_t}dZ_t^P \]  

(15)

with \( T_t = \int_0^t v_s ds \). Define \( \kappa^P \equiv \kappa - \pi_{v} \). In equations (14) and (15), the term \( \pi_{W}T_t \) is the risk premium for the diffusion, the term \( \pi_{X}(k^P_X(1) - k_X(1))T_t \) is the risk premium for the jump process and \( \pi_{v}v_t dt \) is the risk premium for the volatility. Under the measure change, \( W_t^P \) and \( Z_t^P \) are still Brownian motions. To guarantee the absolute continuity between \( X_t \) and \( X_t^P \), the coefficients \( \alpha \) and \( c \) should keep unchanged and only tail parameters could be different (Sato, 1999; Cont and Tankov, 2004). Thus, under the objective measure, the tempered stable process has the Lévy density with the same structure as that under the risk-neutral measure, but with different tail parameters. Furthermore, we assume that risk-neutral measure is just an exponential tilting of objective measure. This is justified by well-known Esscher transform. Esscher transform is a minimum entropy change of measure method, which indicates that there exist a constant \( \xi \) such that the objective Lévy density is related with the risk-neutral one through

\[ v^P(x) = e^{\xi x}v(x). \]

We then have the objective Lévy density of \( X_t^P \) as

\[ v^P(x) = c \frac{e^{-(\lambda_+ - \xi)x}}{x^{1+\alpha}} 1_{x>0} + \frac{e^{-(\lambda_- + \xi)|x|}}{|x|^{1+\alpha}} 1_{x<0}. \]  

(16)

The intuition behind this measure change is consistent with our understanding of financial market movement. It is large jumps that play very important roles in option pricing and risk management since they influence tail behavior of returns distribution.

For the estimation of model, we need the joint conditional characteristic function of return process and variance rate process under the objective measure. The following proposition gives the tractable joint CCF of log return and variance rate.

**PROPOSITION 2**: The joint conditional characteristic function of log return and variance rate with specifications of (14) and (15) under the objective measure is given by

\[ \phi_{R,v}(u_1, u_2; \tau, v_t) = E^P[e^{iu_1R_{t+\tau} + iu_2\nu_{t+\tau}} | G_t] \]

\[ = e^{iu_1(\tau - \rho + A(u_1, u_2, \tau)) + B(u_1, u_2, \tau)v_t}, \]  

(17)

\[ A(u_1, u_2, \tau) = \frac{\kappa \theta (ac - d)}{bcd} \log \left( \frac{c + de^{br}}{c + d} \right) + \frac{\kappa \theta \tau}{c}, \]  

(18)

\[ B(u_1, u_2, \tau) = \frac{1 + ae^{br}}{c + de^{br}}, \]  

(19)
where

\[ a = iu_2(d + c) - 1, \]
\[ b = \frac{d(-\kappa^P - 2uc) + a(-\kappa^P c + \sigma^2)}{ac - d}, \]
\[ c = -\frac{\kappa^P + \sqrt{\left(\kappa^P\right)^2 + 2\sigma^2u}}{2u}, \]
\[ d = \left(1 - iu_2c\right)\frac{-\kappa^P + iu_2\sigma^2 + \sqrt{\left(\kappa^P\right)^2 + 2\sigma^2u}}{-2iu_2\kappa^P + (iu_2\sigma)^2 - 2u}, \]
\[ u = \varphi^P_W(u_1) + \varphi^P_X(u_1) - iu_1(\pi_W + \pi_X), \]
\[ \varphi^P_W = \frac{1}{2}(iu_1 + u_1^2), \]
\[ \varphi^P_X = \psi^P_X(u_1) + iu_1 \kappa^P_X(1), \]
\[ \kappa^P = \kappa - iu_1 \rho \sigma, \]

PROOF: Proof is given in Appendix A. □

By giving different constraints on parameters of this general model, we could study different models. For example if we make \( \alpha \) less than zero, we could study jump-diffusion stochastic volatility model. With these constraints and corresponding models, we could answer the questions brought up in Section 1.

3 Econometric Methodology

When stochastic volatility and jump are introduced to modeling stock price process, it is usually difficult to get an analytical probability density function for the distribution of returns and maximum likelihood estimation can not be applied directly. However, the tractable characteristic function can be obtained easily. Since the characteristic function and the probability density function contain the same information and are equivalent, we could then apply characteristic function directly to implement estimation. We assume that stock market and options market are fully integrated together. It is well-known that the information content in stock market is different from that in options market. The stock market usually contains the historical information about stock price evolution, whereas options market has the expectation information on the future evolution of stock price. Parameter estimate should reflect both information. We thus propose a characteristic function based iterative method for the estimation of models with
aims at making full use of information contained both in stock market and options market and
of tractability of characteristic functions of our models.

3.1 CCF Based Iterative Estimation

For our models, we have two sets of parameters and a sequence of state variable \( v_t \). Two sets of parameters are those of the risk-neutral parameters and the risk premium parameters. We denote them as

\[
\Theta^{RN} = (\kappa, \theta, \sigma, \rho, c, \lambda_+, \lambda_-, \alpha)
\]

and

\[
\Theta^{RP} = (\pi_w, \xi, \pi_v),
\]

respectively. The main assumptions here are that there exists an one-to-one relationship between observed option prices and unobserved variance rates and a fixed point argument could ensure the convergence of our method. Our iterative method works with the following steps:

**Step 1:** Given any initial guess of parameters \( \Theta^{RN}(0) \), we use options data to back out variance rate. Thanks to the analytical characteristic function of return in our model, we could compute option price with fractional fast Fourier transform. In principle, we could use any options which are traded on the market. But when implementing estimation, we only choose at-the-money short maturity call options. This is because these options are the most liquid instruments and convey the most precise information about market fluctuation.

At this step, we obtain a sequence of variance rates \( (v_t^{(1)})_{t=0}^T \).

**Step 2:** With the variance rate \( (v_t^{(1)})_{t=0}^T \) obtained at Step 1 and the stock price data, under the objective measure we implement the joint conditional characteristic function based GMM with a continuum of moment conditions (Joint CCF-CGMM) proposed by Carrasco et al. (2007), which is described in the next subsection. To avoid the identification problem, at this step we conduct an internal iterative loop to iteratively estimate the risk premium parameters and the risk-neutral parameters, that is, firstly conditional on the risk premium parameters, we estimate the risk-neutral parameters; and then conditional on this estimated risk-neutral parameters, we estimate the risk premium parameters. Repeating a certain number of times, we get the risk-neutral parameter estimates \( \Theta^{RN}(1) \) and the risk premium estimates \( \Theta^{RP}(1) \). Here the convergence is very fast.

**Step 3:** With the risk-neutral parameter estimates \( \Theta^{RN}(1) \) obtained at Step 2, we back
out the variance rate \((v_t^{(2)})^T_{t=0}\) again as described in Step 1.

**Step 4:** Repeat Step 1, Step 2 and Step 3 many times until a certain criterion is satisfied.

Finally, we obtain the risk-neutral parameter estimates \((\Theta^{RN})^{(n)}\), the risk premium parameter estimates \((\Theta^{RP})^{(n)}\) and a sequence of variance rate \((v_t^{(n)})^T_{t=0}\). These estimates reflect information contained both in stock market and options market. Under certain regularity conditions, the parameter estimates \((\Theta^{RN})^{(n)}\) and \((\Theta^{RP})^{(n)}\) are consistent and normally distributed and the state variable sequence \((v_t^{(n)})^T_{t=0}\) is the best proxy for the true variance rates (Pan, 2002; Pastorello et al., 2003).

### 3.2 Joint CCF-CGMM

In this subsection, we summarize the recently developed conditional characteristic function based GMM with a continuum of moment conditions (CCF-CGMM; Carrasco et al., 2007) and show how this method could be used to estimate the models studied in this paper. The CCF-CGMM is computationally less demanding than the usual used simulation-based method like EMM (Gallant and Tauchen, 1996) and traditional GMM. It also solves problems of singularity and unstability induced by the discrete moment condition CCF-GMM (Singleton, 2001).

The models in this paper contain unobserved state variable. The (log) stock price process is no longer Markovian. But the full system \(Y_t = (R_t, v_t)\) is a Markov process and stationary. The joint conditional characteristic function has already been derived in Section 2. If we can observe state variable \(v_t\) somehow, we could then implement (two-dimensional) characteristic function based GMM with a continuum of moment conditions. With the definition of conditional characteristic function, we have the following conditional moment conditions

\[
E\left[ \exp\{iuY_{t+\tau}\} - \phi_{R,v}(u; \tau, v_t) \right| G_t] = 0,
\]

where \(u = (u_1, u_2)\). The first term in (20) is the empirical joint characteristic function and the second is the theoretical joint conditional characteristic function of return and variance rate derived in section 2. Since the joint conditional characteristic function of \(R_{t+\tau}\) and \(v_{t+\tau}\) depends only on \(v_t\), the conditional moments (20) could be transformed to unconditional moments if there exists a set of instruments \(Z(\cdot, v_t)\). We then have

\[
E\left[ Z(\cdot, v_t) \left( \exp\{iuY_{t+\tau}\} - \phi_{R,v}(u_1, u_2; \tau, v_t) \right) \right] = 0.
\]
$v_t$ is usually unobservable. However, since there are two markets (stock market and options market) based on the same stock price dynamics, we could then back $v_t$ out from options data and regard it as if it is observable. We thus could apply CCF-CGMM, which extend discrete moment condition CCF-GMM to a continuum of moment conditions with respect to characteristic index and spanning optimial instrument by exponential functions. By doing so, the resulting CCF-CGMM solves problems of efficiency and singularity of covariance matrix. For our problem, the instrument can be constructed by

$$Z(z, X_t) = e^{izv_t},$$  \hspace{1cm} (22)$$

and accordingly we have moment functions

$$\epsilon(u, z; v_t, \beta) = e^{izv_t} \left[ \exp\{iuY_{t+\tau}\} - \phi_{R,v}(u_1, u_2; \tau, v_t) \right],$$  \hspace{1cm} (23)$$

where $\beta$ is a vector of parameters which we are going to estimate. Note that although the optimal instrument can not be obtained, it could be spanned by a set of basis functions (22). Under certain regularity conditions, CCF-CGMM estimation results in ML efficiency if state variable $v_t$ is really observable through:

$$\hat{\beta}_T = \arg \min_{\beta \in B} \left\| \bar{\epsilon}_T(\beta) \right\|_{W_T}^2,$$  \hspace{1cm} (24)$$

where $\bar{\epsilon}_T(\beta) = \frac{1}{T} \sum_{t=1}^T \epsilon(u, z; v_t, \beta)$ is the sample counterpart of moment conditions; $W_T$ is the weighting covariance operator; $\| \cdot \|$ stands for a norm defined in a Hilbert space of complex-valued functions 3 and $B$ is a compact parameter space. The estimator $\hat{\beta}_T$ is asymptotically normal,

$$\sqrt{T}(\hat{\beta}_T - \beta_0) \xrightarrow{d} \mathcal{N}(0, V_T),$$  \hspace{1cm} (25)$$

and

$$V_T = \left\langle E\left( \frac{\partial \bar{\epsilon}_T(\beta)}{\partial \beta} \right), E\left( \frac{\partial \bar{\epsilon}_T(\beta)}{\partial \beta} \right) \right\rangle_{W_T}^{-1},$$  \hspace{1cm} (26)$$

where $\beta_0$ is true parameter and $\langle \cdot, \cdot \rangle_{W_T}$ indicates inner product with respect to $W_T$ in the defined Hilbert space.

In their original paper, Carrasco et al. (2007) deliberately discuss how to construct the weighting covariance operator $W_T$ by introducing a regularization parameter and then sim-
plify the optimization problem by means of the fact that \( \{\epsilon_t\}_{t \geq 0} \) forms a martingale difference sequence with respect to the filtration \( \{S_t\}_{t \geq 0} \). We suggest to simply use identity matrix as weighting covariance operator. In GMM, Cochrane (2005) advocates to use identity matrix as the weighting matrix, which gives more robust estimates. Our Monte Carlo study with stochastic volatility model in the next subsection justifies this selection under CCF-CGMM framework.

### 3.3 A Monte Carlo Study of Joint CCF-CGMM

In this subsection, we conduct a simple Monte Carlo study using Heston’s stochastic volatility model with aims to manifest the estimation efficacy of Joint CCF-CGMM. The Heston’s stochastic volatility model can be obtained from our general model by simply suppressing the tempered stable process and related components. Under the objective measure, it has the form:

\[
S_t = S_0 \exp \left\{ \mu t + \left( W_{T_t} - \frac{1}{2} T_t \right) \right\},
\]

\[
d\nu_t = \kappa (\theta - \nu_t) dt + \sigma \sqrt{\nu_t} dZ_t,
\]

with \( T_t \equiv \int_0^t \nu_s ds \). This time-changed form is equivalent in distribution to the original model (Heston, 1993).

By following the same way as that in Proposition 2, we can derive the objective joint conditional characteristic function of return and variance rate. With this joint CCF, we implement the Joint CCF-CGMM estimation discussed in previous subsection. The weighting covariance operator is selected as identity matrix.

The Monte Carlo study is based on 500 simulations with sample size 500 in weekly frequency. We simulate Heston’s model with an efficient scheme (Andersen, 2007). In this model we have five parameters \( \Theta = (\mu, \kappa, \theta, \sigma, \rho) \) and the true values are given by \( \Theta_0 = (0.150, 6.000, 0.025, 0.300, -0.600) \). Table 1 presents Monte Carlo study results, from which we find Joint CCF-CGMM performs well.

— Table 1 around here —

### 4 Data

The data used in this paper are S&P 500 index and index options traded in Chicago Board Options Exchange (CBOE) over the period from January, 1996 to December, 1999. The data
are in weekly frequency. There are totally 202 weeks. The dataset contains the following series on option Trading Date, Expiration Date, Strike Price, Last Price, Last Bid Price, Last Ask Price and Underlying Price. The interest rates are proxied by US 3-month Treasury bill rates, which, together with dividend yields of S&P 500 index, are downloaded from Datastream.

Figure 1 plots the time-series of S&P 500 index and index returns, from which the characteristics of “jumps” and “time-varying/stochastic volatility” are obvious. For the purpose of estimation of models, we use S&P 500 index prices and index at-the-money short maturity call options. The at-the-money short maturity (ATM-SM) calls are constructed as follows: among all the call options, we choose those with moneyness larger than 0.97 and less than 1.03 and maturity larger than 15 days and less than 45 days. Whenever at each time instant there are more than one call options, we select that with moneyness closest to 1. The constructed ATM-SM calls have mean moneyness 1.000 with standard deviation 0.003 and mean maturity about 25 days with standard deviation 7.3 days. The Black-Scholes implied volatilities, maturity and moneyness are depicted in Figure 2.

We also use call options from June, 1997 to December, 1999 to test models. The following filters are conducted to the dataset: (1) removing the put options from the dataset. We only use call options; (2) we select call options whose maturities are greater than 6 days and less than 1.5 years and whose moneyness less than 1.06; and (3) we exclude call options whose last bid prices are less than 3/8 dollar. By doing so, we obtain a cross sectional call options with 5,793 weekly observations. Whenever the last price of option in dataset is zero, we proxy it with midprice of last bid price and last ask price. Table 2 gives descriptive statistics of S&P index returns, constructed ATM-SM calls and the filtered call options.

5 Results and Discussion

In this section, we present the estimation results and discuss their implications. Subsection 5.1 presents parameter estimates and variance rate estimates; Subsection 5.2 qualitatively discusses the goodness-of-fit of models under CCF-CGMM framework; and Subsection 5.3 further studies model comparison through pricing cross-sectional call options.
5.1 Iterative Joint CCF-CGMM Estimators

Table 3 reports the estimation results including estimates and standard deviation. Models are estimated by Iterative Joint CCF-CGMM described in Section 3. The standard errors are presented in brackets.

We first estimate the general model without any restriction (“LTS-SV” model). The iterative Joint CCF-CGMM is implemented with the number of total iterations 80. In each iteration, we use an iterative approach again to estimate risk-neutral parameters and risk-premium parameters in order to avoid the identification problem. The convergence of this internal iteration is very fast. We give the number of this iteration 10. The initial values, which determine the success or failure of the algorithm, are carefully selected through trying different values. Figure 3 plots the iteration of parameter estimates and their convergence.

Looking at jump related parameters, we find that the tempered stable process in this model acts as an infinite activity process with infinite variation since the estimate of $\alpha$ is 1.132 (positive and larger than one). The estimates of risk-neutral tail parameters $\lambda_+$ and $\lambda_-$ are respectively 22.635 and 1.635, indicating fast right tail dampening and left-skewed distribution. We have discussed in Section 2 that under measure change, only tail parameters change, and the other two ($c$ and $\alpha$) are keep constant. The objective tail parameters are related to risk-neutral ones through $\lambda^P_+ = \lambda_+ - \xi$ and $\lambda^P_- = \lambda_- + \xi$. $\xi$ is a risk-premium parameter. The positive estimate of $\xi$ 8.783 implies that risk-neutral distribution of tempered stable process is more left-skewed than the objective one. Turning to variance rate parameters, the estimate of $\kappa$ 18.673, which controls the speed of mean reversion, indicates that the variance rate quickly reverts to its long-term value $\theta$, which has the estimate of 0.018. The estimate of $\sigma$ is 0.929, which is relatively large. The effect of this large estimate is counteracted by large estimate of $\kappa$. They together determine a moderate variance rate process. We have negative risk premium of volatility $\pi_v$ (-2.992), which is consistent with negative correlation between return process and variance rate process. Our estimate of correlation parameter $\rho$ is -0.934. We observe that the risk premium of diffusion is very tiny, indicating that the market doesn’t take this risk factor into account and only jump and stochastic volatility are priced.

We now study another case in which the tempered stable process behaves like compound
Poisson process. Taking negative value of $\alpha$ in our general model, we obtain compound Poisson type Jump-Diffusion stochastic volatility model but with rich structure. We refer to this model as “LTS-SVJD”, with which we could compare infinite activity stochastic volatility model and jump-diffusion stochastic volatility. We give a value of $\alpha$ -0.01. The parameter estimates of variance rate process do not change much with comparison to those of “LTS-SV” model, but jump related parameter estimates are very different. The estimate of $c$, which reflect jump frequency, has a very large value. It is about 230. Our negative value of $\alpha$ forces $c$ to capture both large jumps and small ones and in this case large jumps and small jumps are indistinguishable. The large $c$ counteracts the effect of small (negative) $\alpha$ such that “LTS-SVJD” model could correctly reflect stock price dynamics and results in the similar values of skewness and kurtosis to those in “LTS-SV” model. The difference of tail parameters $\lambda_+$ and $\lambda_-$ becomes smaller. The risk premium of diffusion in this model is still very small. This seems counterfactual since in Jump-Diffusion model the diffusion part plays more important role than it does in infinite activity model.

The tempered stable process can also take on infinite activity but with finite variation when $\alpha$ is in the range $[0, 1)$. We study this case by simply taking $\alpha$ equal to 0 and thus the tempered stable process becomes Variance Gamma process studied by Madan et al. (1998). The corresponding model is in between our general model and jump-diffusion stochastic volatility model and we refer to it as “LTS-SVVG”. Again, the variance rate parameter estimates are not very different from the LTS-SV model, but jump parameter estimates are different. The estimates are very similar between LTS-SVJD model and LTS-SVVG model since we select a very small value (in absolute) of $\alpha$ in LTS-SVJD model. We find that risk premia of diffusion and volatility are very robust over the models: diffusion premium is negligible and volatility premium is about -3.

We now look at the variance rate estimates. Figure 4 plots the estimated variance rates (square-root) of three models we studied. The general shapes are similar to that of Black-Scholes implied volatility but the values are very different. The means of square-root variance rates of LTS-SV model, LTS-SVJD model and LTS-SVVG model are 12.4%, 11.2% and 11.2% respectively. They are all lower than the mean of Black-Scholes implied volatility. The implication of variance rates is different from that of implied volatility. The variance rates are related not only to the instantaneous variance of diffusion part, but also to the jump arrival rate of tempered stable process. Similarity of variance rates is still observable between LTS-SVJD
model and LTS-SVVG model.

— Figure 4 around here —

5.2 Qualitative Test of Models’ Goodness-of-Fit

In GMM, $J_T$ statistics could be applied to test of model’s goodness of fit, but under the framework of CGMM, we haven’t appropriate statistics to conduct this test since we use infinite number of moment conditions. Therefore, in this subsection we make a qualitative test through checking the discretized moment conditions.

In implementation of CCF-CGMM estimation, we approximate (double) integral with sums in objective function (24) by selecting 16 equally spaced values of each characteristic index ($u_1$ and $u_2$) from the range $[-\pi, \pi]$ and thus we have 256 moment conditions totally. Figure 5 (left panels) plots the real parts of these moment conditions under LTS-SV, LTS-SVJD and LTS-SVVG. We find that the moment conditions have periodic feature and LTS-SV model performs a little better than the other two models. The mean of moment conditions is 1.537 for LTS-SV model, whereas it is 1.785 for LTS-SVJD model and 1.753 for LTS-SVVG model. We have mentioned in Section 3 that since the full system $(\ln S_t, v_t)'$ is Markovian, the sequence of moment functions $\{\epsilon_t\}_{t>0}$ forms martingale difference sequence with respect to $G_t$ and hence it is uncorrelated. Figure 5 (right panels) plots autocorrelations of the first two lags for $\{\epsilon_t\}_{t>0}$ of each discretized moment functions under three models. The means of the first autocorrelation are nearly the same. They are -0.160. Whereas the means of the second autocorrelation are -0.011, -0.014 and -0.014 respectively for LTS-SV, LTS-SVJD and LTS-SVVG.

— Figure 5 around here —

The above discussion implies that the infinite activity stochastic volatility model is more appropriate than jump-diffusion stochastic volatility model in fitting stock price evolution. The discussion of this subsection is intuitive and qualitative. We continue to discuss models’ superiority in next subsection through their performance in pricing options.

5.3 Cross-Section Option Pricing

We begin to compute option prices with parameters and variance rates estimated by Iterative CCF-CGMM method for different models. In practice, this is more interesting. We test models according to their capacity in pricing options. The option prices are computed with FRFT.
We apply two measures to evaluate the models. The first is absolute pricing error, denoted as “Aerr” and the second is relative pricing error, denoted as “Rerr”. These two measures are defined respectively as

\[
Aerr = \frac{1}{N} \sum_{t=1}^{T} \sum_{i=1}^{n_t} |P_{ti}^{im} - P_{ti}|,
\]

(29)

\[
Rerr = \frac{1}{N} \sum_{t=1}^{T} \sum_{i=1}^{n_t} \frac{|P_{ti}^{im} - P_{ti}|}{P_{ti}},
\]

(30)

where \( N \) is the total number of options we consider, \( T \) is the number of weeks, \( n_t \) is the number of options at date \( t \), \( P_{ti}^{im} \) is model-implied option price and \( P_{ti} \) is market option price of \( i \)th option at date \( t \).

Table 4 presents the absolute and relative pricing errors for different models. The following findings are observed: (1) for short maturity call options (maturity less than 60 days), we find that LTS-SV model performs better than the other two models, no matter which moneyness we consider. All three models underprice the options with moneyness less than 1.00, while overprice options with moneyness larger than 1.00; (2) for medium maturity call options (maturity between 60-180 days), LTS-SV model is better than the other two models for those options with moneyness less than 0.94 and larger than 1.03. LTS-SV model underprices all the options except those with moneyness larger than 1.03, whereas the other two overprice the options with moneyness larger than 0.97; (3) for the long maturity call options (maturity larger than 180 days), LTS-SV model performs better than the other two only for options with moneyness in [0.94,0.97] and larger than 1.03. All models overprice the options; (4) LTS-SVVG model and LTS-SVJD are incomparable. They perform nearly the same for all options; (5) whenever LTS-SV underperforms the other two models, its inferiority is very small; and (6) relatively, among all call options, these three models perform best in pricing long maturity in-the-money options and worst in pricing deep out-of-the-money short maturity options.

--- Table 4 around here ---

6 Extension: Double-Jump Model

Eraker et al.(2003) argue in their study of Jump-Diffusion stochastic volatility models with S&P 500 index data that a jump component is also necessary in volatility process. Thus, a natural extension of our general model is to introduce jump in variance rate process.
model the variance rate process under the risk-neutral measure with the following SDE,

\[
dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dZ_t + dJ_t, \tag{31}
\]

where \(Z_t\), as before, is a Brownian motion, which is correlated with \(W_t\) (Brownian motion in return process) and independent of \(X_t\) (tempered stable process in return process). The new process \(J_t\) is a compound Poisson pure jump process, independent of \(W_t\), \(Z_t\) and \(X_t\), whose jump sizes are independent and exponentially distributed with mean \(\mu_J\) and whose jump times follow a Poisson process with jump intensity \(\lambda_J\). The variance rate process (31) is the so-called basic affine process (Duffie and Garleanu, 2001). With this specification, we now have jump components both in return process and variance rate process.

We assume that under change of measure, the jump process \(J_t\) does not change its parameters, that is, the risk premium for risk factor \(J_t\) is zero. Therefore, the implementation of measure change in this Double-Jump model is the same as before. We thus have the objective model as follows,

\[
S_t = S_0 \exp \left\{ (r - q)t + \pi_W T_t + \left[ k_X^P(1) - k_X(1) \right] T_t 
+ \left[ W_{TT}^P - \frac{1}{2} T_t \right] + \left[ X_{TT}^P - k_P^P(1) T_t \right] \right\},
\]

\[
dv_t = [\kappa(\theta - v_t) + \pi_v v_t]dt + \sigma\sqrt{v_t}dZ_t^P + dJ_t, \tag{32}
\]

with \(T_t = \int_0^t v_s - ds\). Define \(\kappa^P \equiv \kappa - \pi_v\) and \(\pi_X \equiv (k_X^P(1) - k_X(1))T_t\). Following the same way as we did in Proposition 2, we could derive the joint conditional characteristic function of return process and variance rate under the objective measure (also see Duffie and Garleanu, 2001).

**PROPOSITION 3:** The joint conditional characteristic function of log return and variance rate with specifications of (32) and (33) under the objective measure is

\[
\phi_{R,v}(u_1, u_2; \tau, v_t) = E^P[e^{iu_1 R_t + \tau + iu_2 v_t + \tau}|G_t]
= e^{iu_1(r-q)\tau + A(u_1, u_2, \tau) + B(u_1, u_2, \tau)v_t}, \tag{34}
\]

\[
A_1(u_1, u_2, \tau) = \frac{\kappa \theta (a_1 c_1 - d_1)}{b_1 c_1 d_1} \log \left( \frac{c_1 + d_1 e^{b_1 \tau}}{c_1 + d_1} \right) + \frac{\kappa \theta}{c_1} \tau, \tag{35}
\]

\[
A_2(u_1, u_2, \tau) = \frac{\lambda_J (a_2 c_2 - d_2)}{b_2 c_2 d_2} \log \left( \frac{c_2 + d_2 e^{b_2 \tau}}{c_2 + d_2} \right) + \frac{\lambda_J(1 - c_2)}{c_2} \tau, \tag{36}
\]

\[
B(u_1, u_2, \tau) = \frac{1 + a_1 e^{b_1 \tau}}{c_1 + d_1 e^{b_1 \tau}}, \tag{37}
\]
where \( A(u_1, u_2, \tau) = A_1(u_1, u_2, \tau) + A_2(u_1, u_2, \tau) \) and

\[
\begin{align*}
A_1 &= iu_2(d_1 + c_1) - 1, \\
b_1 &= d_1(-\kappa P^* - 2u c_1) + a_1(-\kappa P^* c_1 + \sigma^2), \\
c_1 &= -\frac{\kappa P^* \sqrt{2}}{2 \sigma^2} + \frac{(\kappa P^*)^2 + 2 \sigma^2 u}{2 u}, \\
d_1 &= (1 - iu_2 c_1) - \frac{\kappa P^* + iu_2 \sigma^2 + \sqrt{2}}{2 \sigma^2} \frac{(\kappa P^*)^2 + 2 \sigma^2 u}{2 u}
\end{align*}
\]

\[
\begin{align*}
a_1 &= d_1, \\
b_2 &= b_1, \\
c_2 &= 1 - \frac{\mu J}{\lambda J}, \\
d_2 &= d_1 - \mu J a_1, \\
u &= \varphi^P_W(u_1) + \varphi^P_X(u_1) - iu_1 (\pi^W + \pi^X), \\
\varphi^P_W &= \frac{1}{2} (iu_1 + u_1^2), \\
\varphi^P_X &= \psi^P_X(u_1) + iu_1 k^P_X(1), \\
\kappa P^* &= \kappa P - iu_1 \rho \sigma.
\end{align*}
\]

**PROOF:** The proof is similar to that of PROPOSITION 2 except that now we solve ODEs

\[
\begin{align*}
\dot{B}(t) &= u + \kappa P^* B(t) - \frac{1}{2} \sigma^2 B^2(t), \\
\dot{A}(t) &= -\kappa \theta B(t) - \frac{\mu J B(t)}{1 - \mu J B(t)},
\end{align*}
\]

with \( A(t) = A(u_1, u_2, \tau) \) and \( B(t) = B(u_1, u_2, \tau) \) as well as boundary conditions \( B(t + \tau) = iu_2 \) and \( A(t + \tau) = 0 \). \( \Box \)

By setting \( u_2 = 0 \) and suppressing the risk-premium parameters, we could obtain risk-neutral conditional characteristic function of return easily. With these characteristic functions in hand, we implement the Iterative Joint CCF-CGMM estimation.

Table 5 presents the parameter estimates of Double-Jump model. Focusing on jump parameters of variance rate process, the mean of jump size \( \mu J \) is 0.044 and the jump intensity \( \lambda J \) is 0.383. The values of \( \mu J \) and \( \lambda J \) indicate that variance rate doesn’t undergo large and frequent jump. To compare with LTS-SV model, we find that introducing jump in variance rate process causes the large jump frequency, controled by \( c \), decreasing and small jump frequency, controled by \( \alpha \), increasing. The smaller value of \( \lambda J \) implies even heavier left tail. The risk-premium parameters nearly keep the same as those of LTS-SV model.
Looking at the moment conditions and autocorrelation of moment functions, we find that the mean of moment conditions is 1.465 and the means of the first two autocorrelation are -0.160 and -0.009. They are all smaller (in absolute) than those of LTS-SV model, indicating better goodness-of-fit. As for other models, we investigate the performance of Double-Jump model in pricing options. Table 4 reports the option pricing errors of Double-Jump model with model name LTS-SVDJ. The improvement in absolute option pricing error over LTS-SV model is observable for short maturity options, but for the medium and long maturity options, it is indistinguishable or even worse. We could conclude that the Double-Jump model we have studied only has marginal improvement in option pricing and we also conclude that unlike double-jump jump-diffusion stochastic volatility model (Duffie, Pan and Singleton, 2000), once the stock price process is modeled with infinite activity Lévy process, Poisson jumps in stochastic volatility is not critically important.

7 Conclusion

We have studied stock price dynamics by taking into account of both stochastic volatility and jump. Jump is mimiced by the tempered stable process, which could be a finite activity process or an infinite activity process depending on the value of its stable index and stochastic volatility is introduced by time changing the stochastic process. For estimation of models, we propose a characteristic function based iterative estimation method, which overcomes the problem of non-tractable probability density functions of our models and eases computational difficulty related to simulation based estimation method and traditional GMM method. Our qualitative test and option pricing performance of models verify our hypothesis that basically the infinite activity stochastic volatility model is more preferrable than jump-diffusion stochastic volatility model. We also make an extension of our general model by introducing jump in variance rate process to investigate double-jump model. The direct use of characteristic function makes it feasible to estimate this double-jump model. The Double-Jump model only has marginal improvement in option pricing. Even though the improvement of the Double-Jump model over the general model is observed for short and medium maturity options, it is undetectable for long maturity options. In infinite activity Lévy models, Poisson jumps in stochastic volatility play less im-
important roles than in double-jump jump-diffusion stochastic volatility models (Duffie, Pan and Singleton, 2000).

In this paper, we estimate the models using stock price data and only at-the-money short maturity call options. And we discard any other call options in estimation. Even though at-the-money short maturity options are the most liquid financial securities, there are many other options which are also very liquid and contain rich information about financial market movement. So we need more powerful tools to figure out the volatility of financial market by taking into account of the larger number of options, not only at-the-money options, but also out-of-the-money and in-the-money options. One of these tools is filtering techniques recently developed in Engineering like sigma-point (particle) filtering. This is one of our future research directions.

**APPENDIX**

**A  Proofs of Proposition 1 and Proposition 2**

We first prove Proposition 2. Since Brownian motion $W^P_t$ and $Z^P_t$ are correlated, we use approach proposed by Carr and Wu (2004) to implement a measure change in order to internalize this correlation. Define a new measure $M$, which is absolutely continuous with respect to objective measure $P$,

$$
\left. \frac{dM}{dP} \right|_{\mathcal{G}_t} = \exp \left\{ \left[ iu(W^P_{T_t} - \frac{1}{2} T_t) + \varphi^P_{W} T_t \right] + \left[ iu(X^P_{T_t} - k^P_X(1) T_t) + \varphi^P_X T_t \right] \right\}. \tag{38}
$$

Under this new measure $M$, the activity rate process becomes,

$$
dv_t = (\kappa \theta - \kappa^* v_t + iu \rho \sigma v_t) dt + \sigma \sqrt{v_t} dZ^M_t
$$

$$
= (\kappa \theta - \kappa^* v_t) dt + \sigma \sqrt{v_t} dZ^M_t, \tag{39}
$$

where $\kappa^* = \kappa - iu \rho \sigma$ and $Z^M_t$ is now independent of $W^P_t$. The joint conditional characteristic function of $R_{t\tau}$ and $u_{t\tau}$ then can be calculated as follows with the approaches proposed by Duffie, Pan and Singleton (2000).
\[ \phi_{R,u}(u_1, u_2; \tau, \upsilon_t) \equiv E^P[e^{iu_1Rt+iuv_{1,\tau}}|\mathcal{G}_t] \]
\[ = e^{iu_1(r-q)\tau}E^P[e^{iu_1[(\pi W+\pi X)T_r+(W_r^P-\frac{1}{2}T_r)+(X_r^P-k_f^P(1)T_r)]+iu_2v_{1,\tau}}|\mathcal{G}_t] \]
\[ = e^{iu_1(r-q)\tau}E^P[M(\tau)e^{-i[\varphi_W^P+\varphi_X^P-iu_1(\pi W+\pi X)T_r+iuv_{2,\tau}]}|\mathcal{G}_t] \]
\[ = e^{iu_1(r-q)\tau}E^M[e^{-uT_r+iuv_{2,\tau}}|\mathcal{G}_t] \]
\[ = e^{iu(r-q)\tau}e^{\varphi_t(1)\upsilon_1}e^{A(u_1,u_2,\tau)+B(u_1,u_2,\tau)\upsilon_1}, \quad (40) \]

where \( u = \varphi_W^P + \varphi_X^P - iu_1(\pi W + \pi X) \) and \( E^M \) is expectation operator under new measure \( M \) and \( A(u_1, u_2, \tau) = A(t) \) and \( B(u_1, u_2, \tau) = B(t) \) solve the following ODEs

\[ \dot{B}(t) = u + \kappa^P B(t) - \frac{1}{2}\sigma^2 B^2(t) \quad (41) \]
\[ \dot{A}(t) = -\kappa \theta B(t), \quad (42) \]

with boundary conditions \( B(t+\tau) = iu_2 \) and \( A(t+\tau) = 0 \). To solve these two ODEs, we obtain

\[
A(u_1, u_2, \tau) = \frac{\kappa \theta (ac-d)}{bcd} \log \left( \frac{c + de^{\kappa \theta \tau}}{c + d} \right) + \frac{\kappa \theta}{c} \tau, \\
B(u_1, u_2, \tau) = \frac{1 + ac^{-\kappa \theta \tau}}{c + de^{\kappa \theta \tau}}, \\
a = iu_2(d + c) - 1, \\
b = \frac{d(-\kappa^P - 2uc) + a(-\kappa^P c + \sigma^2)}{ac - d}, \\
c = -\frac{\kappa^P + \sqrt{(\kappa^P)^2 + 2\sigma^2 u}}{2u}, \\
d = \frac{(1 - iu_2 c)(-\kappa^P + iu_2 \sigma^2 + \sqrt{(\kappa^P)^2 + 2\sigma^2 u})}{-2iu_2 \kappa^P + (iu_2 \sigma)^2 - 2u}, \\
u = \varphi_W^P(u_1) + \varphi_X^P(u_1) - iu_1(\pi W + \pi X), \\
\varphi_W^P(u_1) = \frac{1}{2}(i u_1 + u_1^2), \\
\varphi_X^P(u_1) = \psi_X^P(u_1) + iu_1 k_X^P(1), \\
\kappa^P = \kappa^P - iu_1 \rho \sigma. \]

Under change of measure defined in the text, the result of Proposition 1 can be easily obtained by setting \( u_2 = 0 \) and suppressing the risk-premium parameters.
References


Notes

1. The characteristic function of a Lévy process can be derived from the Lévy-Kintchine theorem:

\[ \psi_X(u) = -i\mu u + \frac{1}{2}u^2 \sigma^2 + \int_{-\infty}^{\infty} (1 - e^{ixu} + iux\mathbf{1}_{|x|<1}) \nu(x) dx, \]

where \( \psi \) is the characteristic exponent and \((\mu, \sigma, \nu)\) is the characteristic triplet.

2. Another process for \( \nu(t) \) is the positive Ornstein-Uhlenbeck process (Barndorff-Nielson and Shephard, 2000),

\[ d\nu(t) = -\lambda \nu(t) dt + dZ_t, \]

where Lévy process \( Z_t \) is a subordinator. Cont and Tankov (2004) point out that even though the trajectories of CIR process and positive OU process are extremely different, the distributions of business time generated from these two processes are nearly the same. In their demonstration, \( Z_t \) is taken to be a gamma process.

3. The estimation is implemented in the Hilbert space of complex-valued functions, which is defined as

\[ L^2(p) = \{ f : \mathbb{R}^d \rightarrow \mathbb{C}; \int |f(u)|^2 p(u) du < \infty \}, \]

where \( p \) is the reference pdf of a distribution and \( p(u) > 0 \) for all \( u \in \mathbb{R}^d \). \( p \) dampens off all the oscillating behavior of integrands in estimation. As long as \( p > 0 \), the choice of \( p \) does not affect the estimation efficiency in large sample. The inner product on \( L^2(p) \) is \( \langle f, g \rangle = \int f(u) \overline{g(u)} p(u) du \) and norm \( ||f|| = \sqrt{\langle f, f \rangle} \), where the overline denotes complex conjugate.

4. There is also information on Volume, High Price, Low Price and Open Price. In this paper we do not use these information.

5. Moneyness is defined as the ratio of underlying price and strike, \( S/K \).

6. We adopt 252-day/50-week year.

7. In estimation \( \alpha \) has a tendency to zero if we do not give a specific value. To make the model as efficient as possible, we simply take a small value (in absolute), say, -0.01.

8. For tempered stable process, its second, third and (excess) fourth central moment are respectively

\[ m_2 = c\Gamma(2-\alpha)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}), \]
\[ m_3 = c\Gamma(3-\alpha)(\lambda_+^{\alpha-3} - \lambda_-^{\alpha-3}), \]
\[ m_4 = c\Gamma(4-\alpha)(\lambda_+^{\alpha-4} + \lambda_-^{\alpha-4}), \]

with which we could calculate skewness and kurtosis.

9. All these values of mean moment conditions are with scale of \( 10^{-6} \).

10. The characteristic function of this jump process \( J_t \) is

\[ \phi_J(z) = \exp \left\{ -t\lambda_J \left( \frac{iz\mu_J}{i z \mu_J - 1} \right) \right\}, \]

where \( z \in \mathbb{R} \) being characteristic index.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>µ</th>
<th>κ</th>
<th>θ</th>
<th>σ</th>
<th>ρ</th>
</tr>
</thead>
<tbody>
<tr>
<td>True Value</td>
<td>0.150</td>
<td>6.000</td>
<td>0.025</td>
<td>0.300</td>
<td>-0.600</td>
</tr>
<tr>
<td>Mean</td>
<td>0.152</td>
<td>6.351</td>
<td>0.026</td>
<td>0.290</td>
<td>-0.625</td>
</tr>
<tr>
<td>Median</td>
<td>0.154</td>
<td>6.424</td>
<td>0.025</td>
<td>0.295</td>
<td>-0.596</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.050</td>
<td>1.824</td>
<td>0.008</td>
<td>0.061</td>
<td>0.152</td>
</tr>
</tbody>
</table>

**Note:** Monte Carlo Study is conducted with Heston’s SV model,

\[
S_t = S_0 \exp \left( \mu t + \left( \frac{W_{T_1}}{2} - \frac{T_1}{2} \right) \right), \\
\text{d}v_t = \kappa(\theta - v_t)dt + \sigma \sqrt{v_t} \text{d}Z_t.
\]

The number of simulations is 500 with sample size 500 in weekly frequency. Simulation is implemented with algorithm described in Appendix C.
### Table 2: Descriptive Statistics of Data

#### A. S&P 500 Index Returns

<table>
<thead>
<tr>
<th></th>
<th>T</th>
<th>Mean</th>
<th>St. Dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>JB Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weekly</td>
<td>202</td>
<td>0.215</td>
<td>0.174</td>
<td>-0.837</td>
<td>6.668</td>
<td>1(&lt; 0.001)</td>
</tr>
</tbody>
</table>

#### B. Constructed ATM-SM Calls

<table>
<thead>
<tr>
<th></th>
<th>Mean Mn.</th>
<th>Std Mn.</th>
<th>Mean Mt.</th>
<th>Std Mt.</th>
<th>Mean IV.</th>
<th>Std IV.</th>
</tr>
</thead>
<tbody>
<tr>
<td>ATM-SM</td>
<td>1.000</td>
<td>0.003</td>
<td>25.114</td>
<td>7.328</td>
<td>0.187</td>
<td>0.054</td>
</tr>
</tbody>
</table>

#### C. S&P 500 Index Call Options

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Moneyness S/K</th>
<th>&lt; 0.94</th>
<th>0.94-0.97</th>
<th>0.97-1.00</th>
<th>1.00 -1.03</th>
<th>1.03 -1.06</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short</td>
<td>&lt; 60</td>
<td>3.531</td>
<td>14.502</td>
<td>25.895</td>
<td>43.515</td>
<td>68.664</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(449)</td>
<td>(391)</td>
<td>(1016)</td>
<td>(1112)</td>
<td>(778)</td>
</tr>
<tr>
<td>Medium</td>
<td>60-180</td>
<td>18.235</td>
<td>43.745</td>
<td>61.060</td>
<td>83.623</td>
<td>106.647</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(619)</td>
<td>(233)</td>
<td>(199)</td>
<td>(182)</td>
<td>(159)</td>
</tr>
<tr>
<td>Long</td>
<td>&gt; 180</td>
<td>32.670</td>
<td>88.464</td>
<td>109.020</td>
<td>127.213</td>
<td>147.568</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(247)</td>
<td>(118)</td>
<td>(100)</td>
<td>(98)</td>
<td>(92)</td>
</tr>
</tbody>
</table>

*Note:* Table presents descriptive statistics of data we use for model estimation and option pricing. Data are from Jan., 1996 to Dec., 1999 in weekly frequency for estimation. There are totally 202 weeks. In panel A, JB Test is Jarque-Bera normality test, whose value 1 means the test rejects the hypothesis of normality; In panel B, Mn stands for moneyness and Mt maturity (in days). For option pricing, we use option data from Jun., 1997-Dec., 1999. There are totally 5,793 call options. Panel C presents the mean price and the number of call options (in bracket) in each group. We consider totally 15 groups.
## Table 3: Iterative Joint CCF-CGMM Estimation of Models

<table>
<thead>
<tr>
<th>Model</th>
<th>Risk-Neutral Parameters</th>
<th>Risk-Premium Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\kappa$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>LTS-SV</td>
<td>18.673</td>
<td>0.018</td>
</tr>
<tr>
<td></td>
<td>(0.792)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>LTS-SVJD</td>
<td>18.956</td>
<td>0.015</td>
</tr>
<tr>
<td></td>
<td>(1.574)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>LTS-SVVG</td>
<td>18.970</td>
<td>0.015</td>
</tr>
<tr>
<td></td>
<td>(1.274)</td>
<td>(0.003)</td>
</tr>
</tbody>
</table>

*Note:* Models are estimated with Iterative Joint CCF-CGMM discussed in Section 3. Standard deviations are reported in brackets. LTS-SV is our general model; LTS-SVJD is jump-diffusion type stochastic volatility model with $\alpha$ in tempered stable process given by -0.01; LTS-SVVG is the model with $\alpha$ equal to 0 in tempered stable process.
### Table 4: Absolute and Relative Option Pricing Errors

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Model</th>
<th>Absolute Error (Aerr)</th>
<th>Relative Error (Rerr)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>&lt; 0.94</td>
<td>0.94-0.97</td>
</tr>
<tr>
<td>&lt; 60</td>
<td>LTS-SV</td>
<td>0.442</td>
<td>1.148</td>
</tr>
<tr>
<td></td>
<td>LTS-SVJD</td>
<td>0.608</td>
<td>1.635</td>
</tr>
<tr>
<td></td>
<td>LTS-SVVG</td>
<td>0.607</td>
<td>1.635</td>
</tr>
<tr>
<td></td>
<td>LTS-SVDJ</td>
<td>0.435</td>
<td>1.139</td>
</tr>
<tr>
<td></td>
<td>LTS-SV</td>
<td>1.341</td>
<td>1.932</td>
</tr>
<tr>
<td></td>
<td>LTS-SVJD</td>
<td>1.482</td>
<td>1.901</td>
</tr>
<tr>
<td></td>
<td>LTS-SVVG</td>
<td>1.478</td>
<td>1.898</td>
</tr>
<tr>
<td></td>
<td>LTS-SVDJ</td>
<td>1.337</td>
<td>1.944</td>
</tr>
<tr>
<td>60-180</td>
<td>LTS-SV</td>
<td>1.857</td>
<td>2.388</td>
</tr>
<tr>
<td></td>
<td>LTS-SVJD</td>
<td>1.755</td>
<td>2.423</td>
</tr>
<tr>
<td></td>
<td>LTS-SVVG</td>
<td>1.758</td>
<td>2.420</td>
</tr>
<tr>
<td></td>
<td>LTS-SVDJ</td>
<td>1.846</td>
<td>2.380</td>
</tr>
</tbody>
</table>

**Note:** Absolute pricing error and relative pricing error are calculated with formula (29) and (30) respectively. The data for option pricing are from June, 1997 to December, 1999 and they are described in Section 4 and Table 2. Aerr is in dollar. The signs “+” and “-” in brackets represent overpricing and underpricing averagely of each model for each group of call options.
Table 5: **Parameter Estimates of Double-Jump Model**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>VR CIR</th>
<th>VR Jump</th>
<th>Return Jump</th>
<th>Risk Premia</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>18.526</td>
<td>$\mu_J$</td>
<td>0.044</td>
<td>c</td>
</tr>
<tr>
<td></td>
<td>(0.662)</td>
<td>(0.005)</td>
<td></td>
<td>(0.683)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.017</td>
<td>$\lambda_J$</td>
<td>0.383</td>
<td>$\lambda_+$</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.037)</td>
<td></td>
<td>(2.112)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.863</td>
<td></td>
<td>$\lambda_-$</td>
<td>0.380</td>
</tr>
<tr>
<td></td>
<td>(0.028)</td>
<td></td>
<td></td>
<td>(0.171)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.999</td>
<td>$\alpha$</td>
<td>1.364</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.020)</td>
<td></td>
<td></td>
<td>(0.119)</td>
</tr>
</tbody>
</table>

*Note:* Parameter estimates of double-jump model with Iterative Joint CCF-CGMM. VR CIR represents the parameters related to CIR part in variance rate (31); VR Jump represents the parameters in jump process $J_t$; Return Jump represents the parameters in tempered stable process and Risk Premia represents risk-premium parameters. Standard deviations are in brackets.
Figure 1: Time Series of S&P 500 Index and Index Returns
Figure 2: Implied Volatility, Maturity and Moneyness of ATM-SM Call Options
Figure 3: The Convergence of Iterative Joint CCF-CGMM Estimation in LTS-SV Model
Figure 4: Estimated Square-Root Variance Rates
Figure 5: Moment Conditions and Autocorrelation of Moment Functions
Figure Legends:

Figure 1: Time Series of S&P 500 Index and Index Returns. S&P index and index returns. Upper panel plots time series of Index prices and lower panel plots time series of index returns.

Figure 2: Implied Volatility, Maturity and Moneyness of ATM-SM Call Options. Upper panel is the Black-Scholes implied volatilities of constructed at-the-money short maturity call options; middle panel and bottom panel plot the maturity (in days) and moneyness ($S/K$) of ATM-SM respectively.

Figure 3: The Convergence of Iterative Joint CCF-CGMM Estimation in LTS-SV Model. Figure plots the parameter convergence in LTS-SV model. The total number of iterations is 80. In each iteration, there are 10 internal iterations for estimation of risk-neutral parameters and risk-premium parameters alternately.

Figure 4: Estimated Square-Root Variance Rates. Figure plots the estimated variance rates for LTS-SV model, LTS-SVJD model and LTS-SVVG model. They are backed out from ATM-SM call options with parameter estimates in Table 3.

Figure 5: Moment Conditions and Autocorrelation of Moment Functions. The left panels plot discretized moment conditions for LTS-SV model, LTS-SVJD model and LTS-SVVG model. There are 256 discretized moment conditions. The right panels plot autocorrelations of the first two lags of each time series of moment functions. The solid lines are the first lag autocorrelation and the dotted-dashed lines are the second lag autocorrelation.