

**A Dynamic Equilibrium Model of Real Exchange Rates
with General Transaction Costs***

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ABSTRACT

The inability of current theoretical research on real exchange rates to reconcile the observed large short-term volatility of the real exchange rate with its slow convergence to parity constitutes the purchasing power parity (PPP) puzzle. We propose a dynamic general equilibrium model of real exchange rates with two countries and one good. The model incorporates a general cost structure for goods transportation between the two countries, capturing the realistic feature that while the total cost increases with the amount of transportation, the unit cost decreases with the shipping volume due to economies of scale. The model endogenizes both the short term real exchange rate volatility and its slow decay to parity, thus resolving the PPP puzzle. Furthermore, we find that the drift and diffusion of the real exchange rate are not uniquely determined by the real exchange rate level. We discuss the implications of this result on future empirical studies.

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A large body of literature has been devoted to the empirical studies of the real exchange rate and deviations from purchasing power parity (PPP). Among many inconclusive and often conflicting findings, researchers have come to a broad consensus on two main observations. First, the real exchange rate converges to purchasing power parity in the very long run, but the speed of convergence is extremely slow and nonlinear in the exchange rate level. Second, the short-run deviations from purchasing power parity are large and volatile. Reconciling the enormous short term volatility of the real exchange rate with its extremely slow convergence to parity represents a challenge to theoretical research. Rogoff (1996) provides an excellent review of this PPP puzzle and argues that any resolution of the PPP puzzle must recognize that the international capital markets are quite segmented, with large trading frictions due to transportation costs, tariff and nontariff barriers, information costs, and labor immobility. Our paper reconciles the high short-term volatility of the real exchange rate with its extremely slow rate of decay to parity in a segmented international goods market with trading frictions due to transportation costs.

In resolving the PPP puzzle, both monetary and real economy models have been proposed in the literature. For example, the overshooting model of Dornbusch (1976) attributes the short term deviations from purchasing power parity to stickiness in nominal prices. Other explanations of the short term exchange rate volatility in monetary models include financial factors such as changes in portfolio preferences, short-term asset price bubbles, and monetary shocks (Obstfeld and Rogoff 1995), but such models cannot generate the observed slow convergence to PPP. On the other hand, as Rogoff (1996) points out, real models can readily explain the slow adjustment of the real exchange rate to parity. Nevertheless, earlier real-economy based models cannot account for the high short-term real exchange rate volatility away from parity. Furthermore, a valid explanation of short-term exchange rate volatility cannot rely too heavily on non-traded goods or institutional factors such as exchange rate regimes because the evidence on high volatility for PPP deviations even among highly traded goods is remarkably stable over the past 700 years (Froot, Kim, and Rogoff (1995)).

This article proposes a dynamic general equilibrium model of real exchange rates with two countries and one good along the lines of Dumas (1992). The model endogenizes both the short term real exchange rate volatility and its slow decay to parity by introducing a general cost structure for goods transportation between the two countries. Specifically, the cost structure captures the realistic feature that while the total cost increases with the amount of transportation, the unit cost decreases with the shipping volume due to, for example, economies of scale. Economies of scale are evident in most aspects of domestic and international trade. For example, economies of scale can arise from fixed non-tradable input costs. In our model, the cost of the vessel for transporting goods is a figurative example that generates such economies of scale.

In the presence of economies of scale in goods transportation, the shipping strategy constitutes the dual decision: when to transfer goods and how much to transfer. We solve the optimization problem and analyze how the structure of transaction cost impacts the optimal shipping strategy and how these impacts carry over in determining the real exchange rate dynamics in equilibrium. We find that introducing even a small degree of economies of scale in the cost structure significantly increases the optimal shipping quantity. Further, including economies of scale renders the optimal shipping strategy more sensitive to production technology and consumer preference.

The impact of the transaction cost structure on the optimal shipping decision also carries over in determining the dynamics of real exchange rates in equilibrium. In particular, we find that increasing the overall cost of goods transportation slows down the speed of reversion of the real exchange rate to parity and increases its volatility. However, the persistence and volatility of the real exchange rate are much more sensitive to the economies of scale component than to the proportional cost component. Furthermore, we find that economies of scale in the cost structure are indispensable to generating large volatility in the real exchange rate *away* from parity.

The presence of economies of scale results in finite goods transfer and hence generates jumps in the dynamic process for capital stock imbalance. Yet, in equilibrium, the real exchange rates before and after each jump in the capital stock must remain the same to exclude arbitrage. The interesting consequence is that two different levels of capital stock imbalance correspond to the same level of real exchange rate. While each capital imbalance level uniquely determines a real exchange rate level, the opposite is not always true. One cannot always infer the capital imbalance from the real exchange rate

level. This loss of one-to-one mapping also generates indeterminacy in the drift and diffusion functions of the real exchange rate dynamics. At a given real exchange rate level, both the drift (mean-reverting force) and the diffusion (instantaneous volatility) of the exchange rate can take either of two values, determined by the capital imbalance at that instant in time.

Most germane to our paper are the real economy models developed by Dumas (1992), Uppal (1993), Hollifield and Uppal (1997), and Dumas and Uppal (2001)). However, in all these papers, the transportation cost is assumed to be *purely proportional* to the amount of shipment, thus ignoring any potential reduction in unit cost due to economies of scale. Such a cost structure leads to two major counterfactual implications. First, the optimal shipping amount is infinitesimal, regardless of production technology and consumer preference. Second, in equilibrium, the real exchange rate exhibits the largest volatility at parity, but the volatility declines monotonically and approaches zero as the deviation from parity increases. In reality, however, the optimal shipping quantity is always a *finite* amount, and the real exchange rate exhibits enormous short term volatility *away* from parity. We capture both features in our model with a general cost structure for goods transportation.

While this article focuses on the construction of a general equilibrium model of real exchange rate, there is also a parallel strand of literature in the portfolio decision literature based on purely fixed or purely proportional transaction costs in optimal portfolio selection. Examples include Atkinson and Wilmott (1995), Constantinides (1986), Cvitanić (1996), Davis and Norman (1990), Duffie and Sun (1990), Korn (1998), and Morton and Pliska (1995).

The structure of the paper is as follows. The next section sets up the model and delineates the procedure for solving the general equilibrium in the presence of a general cost structure. Section II analyzes the optimal shipping decisions, both when and how much to ship, as a function of the cost structure, the production technology, and the consumer preference. Section III analyzes the optimal consumption decision and the physical imbalance dynamics. Section IV investigates the real exchange rate dynamics under such a model. Section V concludes.

I. A Dynamic Equilibrium Model with Transaction Costs

The model consists of two countries (home and foreign) with identical consumers who can own and trade a single type of consumption good. Consumers can only consume goods within their own country, but can invest in a constant-return-to-scale production process for goods in either country. The capital stocks of goods at time t are K_t and K_t^* at home and abroad, respectively. Goods transportation is costly and hence goods can only be transported a finite number of times within any finite time interval. During quiescent periods absent of goods shipment, the dynamics of the capital stocks are governed by the following stochastic differential equations,

$$\begin{aligned} dK_t &= (\kappa K_t - c_t) dt + \sigma K_t dz_t; \\ dK_t^* &= (\kappa K_t^* - c_t^*) dt + \sigma K_t^* dz_t^*. \end{aligned} \tag{1}$$

where (c_t, c_t^*) are domestic and foreign rates of consumption at time t and (dz_t, dz_t^*) denote independent Wiener processes which proxy for production shocks in each country. Note the symmetry between the two countries as the same constant parameters, $\kappa \in \mathbb{R}, \sigma \in \mathbb{R}^+$, govern the dynamics of the capital stocks of both countries. The basic structure of this model is similar to Dumas (1992).

A. The Cost Structure

In contrast to the earlier literature that focuses on purely proportional cost structures, we specify a more general cost structure. Our cost structure captures the realistic feature that, while the total cost increases with the amount of shipment, the unit cost decreases with shipping due to economies of scale. To illustrate this point and motivate our specification of the cost structure, consider a simple construct under which the goods are transported, literally, through a ship. Transportation costs then include the cost of moving the goods from and to a ship, as well as the cost of building and operating the ship. The cost of moving the goods from and to a ship in general increases with the amount of goods that need to be shipped and hence can be regarded as a variable/proportional cost. In contrast, the construction and maintenance cost of the ship is not directly related to the amount of shipment, but is related to the capacity of the ship. Building a smaller ship will cost less, but probably will also incur a larger variable

cost if more than one round of shipping is needed. Ideally, one wants to build a ship that can handle the expected maximum capacity in an efficient way.

To capture these features in goods transportation, we assume that the total cost of transportation has two components. The first component increases with the amount being transported and hence can be regarded as the variable or proportional cost part. The second component is not directly related to the shipping quantity, but is proportional to the expected maximum transfer (capacity). While the first component captures, for example, the labor cost of moving goods on and off the ship, the second component captures the cost of building a ship with a capacity that can handle the maximum transfer between the two countries.

Formally, let X denote the amount of shipment from the home country to the foreign country, and X_{\max} denotes the expected maximum transfer between the two countries.¹ Then X_{\max} represents the necessary capacity for the shipping vessel. We specify that the total cost (TC) is proportional to both the shipping quantity X and the capacity variable X_{\max} :

$$TC = \alpha X + \beta X_{\max}, \quad \alpha, \beta \in [0, 1]. \quad (2)$$

The parameter restrictions in (2) follow from the requirement that the total loss in transit cannot exceed the maximal amount of shipment. We further argue that, irrespective of the cost structure, the home country consumers do not have the incentive to transfer more than the capital stock imbalance, $X_{\max} = K - K^*$. The proof of this argument is given in Appendix A. Thus, given a shipment of X from home to abroad, the capital stock of the home country reduces to $(K - X)$ while the capital stock of the foreign country increases to

$$K^* + X - TC = (1 + \beta)K^* + (1 - \alpha)X - \beta K.$$

¹Due to the strict symmetry of the problem, we only consider the case when the home capital is greater than the foreign capital, $K > K^*$, and shipment from home to abroad is needed. The transfer from abroad to home in the case of $K^* > K$ can be dealt with analogously.

Another way to look at the cost structure in (2) is to rewrite it in terms of the unit cost (UC), i.e., the cost per transporting one unit of goods,

$$UC \equiv \frac{TC}{X} = \alpha + \beta/\xi, \quad (3)$$

where $\xi \equiv X/X_{\max}$ is the shipping quantity as a fraction of the maximum capacity. We observe that, under our specification, while the total cost increases with the shipping quantity, the unit cost decreases (inversely proportional) with it. The highest efficiency (lowest unit cost) is achieved when the vessel is used to full capacity, i.e. $\xi = 1$. Thus, while the parameter α captures the constant, or proportional part of the shipping cost, the parameter β determines the speed of reduction in unit cost with increased shipping and hence can be regarded as a measure of the economies of scale in the transportation of goods.

We use the simple example of building and operating a ship to illustrate, figuratively, the cost structure for goods transportation and to emphasize that costs are certainly not always proportional to the transaction quantity. Indeed, barriers in international trade exhibit themselves in various different forms, such as transportation costs, tariffs, and non-tariff barriers. See Erzan, Kuwahara, Marchese, and Vossenaar (1989) for a profiling analysis of different types of trade barriers among 50 developing countries. In particular, they find that the exact structure and degree of trade protection varies significantly across both geographical regions and income groups. Gagnon (1989) analyzes the trade flows between the United States and three of its largest trading partners and finds that, though tariff has reduced over the last two decades, non-tariff barriers have increased. A more recent study by Hummels and Skida (2001) reveals similar trends in the evolution of the cost structure. These empirical studies stress that trade barriers can have different forms and these different forms both evolve over time and vary across borders. Thus, when modeling international trade and/or real exchange rate, it is imperative to start with a general cost structure and analyze how different cost components impact the optimal trade decision and, ultimately, how they influence the dynamics of the real exchange rate between countries.

B. The Central Planning Problem

The economy, by assumption, is such that consumers can achieve a Pareto-optimal allocation of consumption. Under such an assumption, the capital market and goods market equilibrium can be replicated by a central planning problem. The welfare function is constructed as an equally weighted average of the individual lifetime utility functions. The equal weight is a result of our assumption of strict symmetry between the two countries, including their respective initial endowments. Implicit prices, which would prevail explicitly in decentralized markets, can then be obtained from the derivatives of the indirect utility function.

Consumers of both countries have the incentive to bring the two countries' stock of goods to balance for reasons of risk sharing. Nevertheless, in the presence of a transportation cost, goods shipment between the two countries can only happen a finite number of times within any finite time interval. Capital stock imbalance can persist for a long period of time. Thus, the central planner's decision is twofold: the optimal consumption plan for both home and foreign consumers and the optimal shipping decision. Let $c(K, K^*)$ and $c^*(K, K^*)$ denote the consumption flow at home and abroad as a function of the capital stocks in the two economies. Let $\Omega(K, K^*)$ denote an admissible shipping strategy, which potentially includes decisions on both when the shipping should occur and how much the shipping quantity should be. The central planning problem is to reach optimal consumption and optimal shipment decisions to maximize the expected utility of aggregate consumption:

$$V(K, K^*) \equiv \max_{c, c^*, \Omega} \mathbb{E}_t \int_t^\infty e^{-\rho(u-t)} \left[\frac{1}{\gamma} c_u^\gamma + \frac{1}{\gamma} (c_u^*)^\gamma \right] du, \quad (4)$$

subject to the dynamics of the capital stock as specified in (1) in the absence of shipment, and subject to the cost structure in (2) when shipping takes place. In (4), $\rho \in \mathbb{R}^+$ represents the time discount factor and $(1 - \gamma) \in \mathbb{R}^+$ captures the relative risk aversion. The instantaneous utility of consumption is identical for consumers of both countries and takes the form of constant relative risk aversion (CRRA):

$$u(c, t) = \frac{1}{\gamma} c_t^\gamma, \quad u(c^*, t) = \frac{1}{\gamma} (c_t^*)^\gamma.$$

The CRRA utility, together with our assumption on the cost structure guarantees that the solution for the indirect utility function $V(K, K^*)$ is homogeneous of degree γ . Furthermore, if X is the optimal amount of shipment from home to abroad for initial conditions (K, K^*) , then $2X$ is the optimal shipment for initial conditions $(2K, 2K^*)$. The same homogeneity applies to the shipment from abroad to home.

C. The Optimal Shipping Decisions

Due to the homogeneity and symmetry of the problem, the optimal decision on when to make a shipment can be thought of as being governed by a cone defined by two symmetric boundaries on the ratio of the home and foreign stock of capital, $K/K^* \in [\lambda_1, 1/\lambda_1]$, $\lambda_1 > 1$. K/K^* defines the capital imbalance between the two countries. Shipping occurs only when the magnitude of the capital imbalance reaches either of the two boundaries. No shipment occurs as long as the capital imbalance is within the cone defined by the two boundaries.

When it is optimal to make a shipment, the optimal shipping quantity depends on the exact structure of the transportation cost. For example, when the cost structure is purely proportional, it is optimal to ship only an infinitesimal amount. In the presence of any economies of scale, however, it is always more beneficial to ship more than an infinitesimal amount so as to reduce the unit cost. Given any finite amount of shipment, the capital imbalance would be moved strictly inside the pre-shipment boundary λ_1 , or $1/\lambda_1$. Let $[\lambda_2, 1/\lambda_2]$ denote the after-shipment capital imbalance boundary. Then this new imbalance λ_2 is determined by the pre-shipment boundary λ_1 , the optimal shipping fraction ξ , and the cost of the shipment. In particular, they are linked by,²

$$\lambda_2 = \frac{\lambda_1 - \xi(\lambda_1 - 1)}{(1 + \beta) + (1 - \alpha)\xi(\lambda_1 - 1) - \beta\lambda_1}.$$

Thus, the optimal shipping decision can now be written as a bivariate decision: $\{\lambda_1, \xi\}$, where λ_1 determines when to ship and ξ determines how much. It can also be rewritten in terms of the capital imbalance before and after the shipment: $\{\lambda_1, \lambda_2\}$.

²Refer to Appendix B for the derivation.

D. Solving the Equilibrium Based on a Recursive Procedure

To solve the equilibrium, we decompose the decision into two parts. First, within the no-shipping cone of $K/K^* \in [\lambda_1, 1/\lambda_1]$, the optimal consumption decision (c_t, c_t^*) can be solved based on the standard Hamilton-Jacobi-Bellman equation, e.g. Merton (1971),

$$0 = \left[\frac{1}{\gamma} c_t^\gamma + \frac{1}{\gamma} (c_t^*)^\gamma \right] + \mathcal{L}V, \quad (5)$$

where $\mathcal{L}V$ denotes the infinitesimal generator of V :

$$\mathcal{L}V = -\rho V + V_1(\kappa K_t - c_t) + V_2(\kappa K_t^* - c_t^*) + \frac{1}{2} V_{11} \sigma^2 K^2 + \frac{1}{2} V_{22} \sigma^2 K^{*2},$$

where V_i is the partial derivative of the value function $V(K, K^*)$ with respect to its i th argument, $i = K, K^*$, and V_{ij} denotes the second partial derivative.

The second part of the decision concerns the optimal shipping decision. At the optimal time to make a shipment, we introduce the following maximum operator to determine the optimal shipping quantity:

$$MV(K, K^*) \equiv \max_X V(K - X, (1 + \beta)K^* + (1 - \alpha)X - \beta K) + \frac{1}{\gamma} (c_t^\gamma + (c_t^*)^\gamma), \quad (6)$$

when $K > K^*$. The case of $K < K^*$ is determined by symmetry. Note that $MV(K, K^*)$ represents the value of the strategy that consists of taking the best immediate action. Since it is not always optimal to act immediately, we have

$$V(K, K^*) \geq MV(K, K^*),$$

where the equality holds only when it is optimal to make a shipment, i.e. when $K/K^* = \lambda_1$. The optimal shipping quantity can then be determined by the optimization problem in (6).

D.1. The evolution of the value function

Within the no-shipping zone, from the Hamilton-Jacobi-Bellman equation in (5), the first order conditions with respect to the optimal consumptions are,

$$0 = c_t^{\gamma-1} - V_1; \quad 0 = (c_t^*)^{\gamma-1} - V_2,$$

from which we obtain the optimal consumptions as a function of the indirect utility function:

$$c_t = V_1^{\frac{1}{\gamma-1}}, \quad c_t^* = V_2^{\frac{1}{\gamma-1}}. \quad (7)$$

Plug in the optimal consumption decisions back to equation (5), we obtain the following partial differential equation,

$$0 = \left(\frac{1}{\gamma} - 1\right) \left(V_1^{\frac{\gamma}{\gamma-1}} + V_2^{\frac{\gamma}{\gamma-1}}\right) - \rho V + V_1 \kappa K + V_2 \kappa K^* + \frac{1}{2} V_{11} \sigma^2 K^2 + \frac{1}{2} V_{22} \sigma^2 K^{*2}. \quad (8)$$

This partial differential equation governs the evolution of the value function. The value function can be solved given boundary conditions derived in the following subsection.

D.2. Boundary conditions

In the presence of a general cost structure as in (3), the optimal shipping decision includes both the optimal shipping time and the optimal shipping quantity. Due to the homogeneity of our setup, the two decisions can be represented by the two boundaries: $\{\lambda_1, \lambda_2\}$, the physical imbalance of the capital stock between home and abroad before and after the shipment. The two boundaries are determined by the following four conditions: (1) symmetry, (2) value matching, (3) smooth pasting, and (4) optimal shipping.

Symmetry: Since the two countries are strictly symmetric, we expect that consumers in the two countries have the same marginal utility when the capital stock is at balance ($K = K^*$):

$$V_1(K, K^*) = V_2(K, K^*) \quad \text{at} \quad K = K^*. \quad (9)$$

Value matching: The value function $V(K, K^*)$ before and after shipment should be identical:

$$V(K, K^*) = V(K - X, (1 + \beta)K^* + (1 - \alpha)X - \beta K) \quad \text{at} \quad K/K^* = \lambda_1. \quad (10)$$

Smooth pasting: The marginal utility before and after shipping should also match each other:

$$V_i(K, K^*) = V_i(K - X, (1 + \beta)K^* + (1 - \alpha)X - \beta K) \quad \text{at} \quad K/K^* = \lambda_1, \quad (11)$$

for $i = 1, 2$.

Optimal shipment: At the boundary $K/K^* = \lambda_1$, the optimal shipment is determined by the optimization problem in (6):

$$MV(K, K^*) \equiv \max_X V(K - X, (1 + \beta)K^* + (1 - \alpha)X - \beta K) + \frac{1}{\gamma} (c_t^\gamma + (c_t^*)^\gamma).$$

Optimal consumption in (7) right before shipping implies

$$\begin{aligned} c_t &= V_1(K, K^*) = V_1(K - X, (1 + \beta)K^* + (1 - \alpha)X - \beta K), \\ c_t^* &= V_2(K, K^*) = V_2(K - X, (1 + \beta)K^* + (1 - \alpha)X - \beta K), \end{aligned}$$

The second equality for each equation is the result of imposing the smooth pasting conditions. We hence have

$$MV(K, K^*) = \max_X \left[V + \frac{1}{\gamma} \left(V_1^{\frac{\gamma}{\gamma-1}} + V_2^{\frac{\gamma}{\gamma-1}} \right) \right],$$

where the indirect value function V and its derivatives are all evaluated at the after-shipping position: $(K - X, (1 + \beta)K^* + (1 - \alpha)X - \beta K)$. The first order condition with respect to X is given by,³

$$0 = - \left[V_1 + \frac{1}{\gamma - 1} \left(V_1^{\frac{1}{\gamma-1}} V_{11} + V_2^{\frac{1}{\gamma-1}} V_{21} \right) \right] + (1 - \alpha) \left[V_2 + \frac{1}{\gamma - 1} \left(V_1^{\frac{1}{\gamma-1}} V_{12} + V_2^{\frac{1}{\gamma-1}} V_{22} \right) \right]. \quad (12)$$

Note that the optimal shipping quantity becomes a degenerating decision when the cost structure is purely proportional as we have $X = 0$ (infinitesimal) and $\lambda_1 = \lambda_2$.

In principle, the value functions and the boundaries can be solved from the partial differential equation in (8) and the four boundary conditions: (9), (10), (11), and (12). However, trying to directly solve the problem from the above conditions is a daunting numerical task. In what follows, we take advantage of the homogeneity and symmetry of the problem and solve the problem via some change of variables.

D.3. Solving the boundaries and the value function via transformation

Recognizing the homogeneity of the value function and the optimal shipment decision, we perform the following change of variables,

$$\omega \equiv \ln K / K^*, \quad I(\omega) \equiv -\gamma \ln K^* + \ln \|V\|, \quad I(-\omega) \equiv -\gamma \ln K + \ln \|V\|, \quad (13)$$

where ω defines the logarithm of the capital stock imbalance ratio and $I(\omega)$ defines a transformed value function at such an imbalance. Under this transformation, the partial differential equation in (8) can be transformed into a second order ordinary differential equation in terms of $I(\omega)$:

$$I''(\omega) = - (I'(\omega))^2 + \gamma I'(\omega) + \frac{1}{\sigma^2 \eta} e^{(\eta-1)I(\omega)} (I'(\omega) \zeta_1(\omega) + (\gamma - I'(\omega)) \zeta_2(\omega)) + \delta, \quad (14)$$

³We drop the argument of the value function V for clarity. Nevertheless, we stress that the arguments are based on the after-shipping positions of the capital stock.

where

$$\begin{aligned}\eta &= \frac{\gamma}{\gamma-1}, \quad \delta = \frac{\rho - \alpha\gamma}{\sigma^2} - \frac{1}{2}(\gamma^2 - \gamma), \\ \zeta_1(\omega) &= \|I'(\omega)\|^{\eta-1} e^{-\omega\eta}, \quad \zeta_2(\omega) = \|\gamma - I'(\omega)\|^{\eta-1}.\end{aligned}$$

The symmetry condition in (9) implies,

$$I'(0) = \gamma/2. \quad (15)$$

From the value matching condition in (10), we can obtain the following link between $I(\omega_1)$ and $I(\omega_2)$:

$$I(\omega_1) = I(\omega_2) + \gamma \ln \frac{1 + \beta + \lambda_1(1 - \alpha - \beta)}{1 + (1 - \alpha)\lambda_2}, \quad (16)$$

where (ω_1, ω_2) are the logarithms of (λ_1, λ_2) . The smooth pasting conditions in (11), on the other hand, yield the solutions for $I'(\omega)$ at the two boundaries ω_1 and ω_2 :

$$I'(\omega_1) = \gamma\lambda_1 \left(\frac{1 - \alpha - \beta(\lambda_1 - 1) / (\lambda_1 - \lambda_2)}{1 + \beta + (1 - \alpha - \beta)\lambda_1} \right), \quad (17)$$

$$I'(\omega_2) = \gamma\lambda_2 \left(\frac{1 - \alpha - \beta(\lambda_1 - 1) / (\lambda_1 - \lambda_2)}{1 + (1 - \alpha)\lambda_2} \right). \quad (18)$$

Finally, the first order condition on the optimal shipping in (12) can be used, together with the second order differential equation in (14), to derive the second order derivative of the value function $I''(\omega)$ at ω_2 :

$$I''(\omega_2) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad (19)$$

where A, B, C are functions of $\omega_2, I'(\omega_2)$, and model parameters. The derivation of (19) as well as the specifications for A, B, C are given in Appendix C.

The boundaries and the transformed value functions $I(\omega)$ at all $\omega \in [-\omega_1, \omega_1]$ can be solved through a shooting approach. We start with a guess of the pair (ω_1, ω_2) . We then solve for $I'(\omega_2)$ from (18) and $I''(\omega_2)$ through (19). The transformed value function $I(\omega_2)$ can then be derived from the ordinary differential equation in (14). Given the transformed value function $I(\omega)$ and its derivatives at ω_2 , we numerically run through the ordinary differential equation to generate the value functions and its derivatives at all other ω 's. The optimal boundaries (ω_1, ω_2) are obtained by matching the marginal

value functions $I'(0)$ and $I'(\omega_2)$ implied from the ordinary differential equation with those computed from the symmetry condition in (15) and the smooth pasting condition in (18). The ordinary differential equation can be solved using standard numerical procedure such as the fourth order Runge-Kutta method.

II. Optimal Shipping Policy

Given model parameters, we can solve the model numerically to obtain the optimal shipping decisions $\{\lambda_1, \xi\}$ and the transformed value functions. In this section, we investigate the impact of different model parameters on the optimal shipping decision. Since a nontrivial optimal shipping decision is a direct result of the presence of a general shipping cost structure, we start with the most important parameters, (α, β) , which govern the cost structure for goods transportation. We analyze how the cost structure changes the decisions on when and how much to ship. We then proceed to analyze the impact of the production parameters and consumer preference parameters on the optimal shipping decision. The optimal shipping decision interacts with the optimal consumption decision to determine the dynamics of the capital stock imbalance (Section III) and, ultimately, the dynamics of the real exchange rate in equilibrium (Section IV).

A. Impact of the Cost Structure

Recall that the unit cost of shipment is specified as $UC = \alpha + \beta/\xi$, where ξ denotes the shipment quantity as a fraction of the maximum capacity, α is the coefficient of proportional cost and β the coefficient of economies of scale. The bigger α is, the faster the total cost increases with the amount of shipment. On the other hand, the bigger β is, the larger the economies of scale, and hence the faster the unit cost reduces with shipping quantity. Figure 1 depicts the impact of the cost structure (α, β) on the optimal shipping decisions. The left panel depicts the impact on the optimal timing of the shipment, which is captured by the outer boundary λ_1 of the capital stock imbalance. The right panel depicts the impact on the optimal shipping quantity as a fraction of the shipping capacity, ξ .

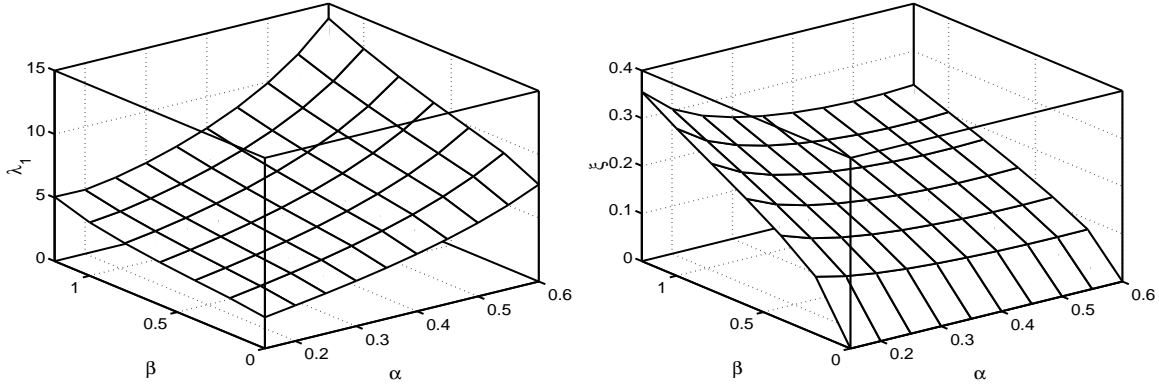


Figure 1. Impact of Cost Structure on Optimal Shipping Decision

The surface in the left panel depicts the optimal shipping boundary, denoted by λ_1 , while the surface in the right panel depicts the optimal shipping quantity as a fraction of the capacity, ξ . Both are plotted against the constant cost coefficient α and the coefficient of economies of scale β . The units of β is scaled up by 10^3 . Other model parameters are: $\kappa = 0.11, \sigma = 0.5, \gamma = -1, \rho = 0.15$.

The left panel indicates that λ_1 increases with both coefficients α and β . As both coefficients α and β increase, the boundary λ_1 , at which shipping occurs, increases. As expected, as the cost increases, regardless of its nature, it is optimal for the consumers to tolerate a larger capital imbalance before making a shipment.

The optimal shipment, ξ , however, is very sensitive to the specific nature of the cost, as shown in the right panel of Figure 1. The optimal shipping quantity increases dramatically with increasing economies of scale (β). Intuitively, the larger β is, the more cost reduction is achieved through shipping a larger amount. In the absence of economies of scale ($\beta = 0$), the unit cost remains the same no matter how much is shipped. Thus, it is optimal to ship just an infinitesimal amount at each time of shipment. But, an infinitesimal amount of shipment is never going to be optimal in the presence of even a very small economies of scale factor. Note that, in the graphs, the magnitude of β is very small (scaled up by 10^3), yet its impact on the optimal shipping amount is very significant.

While increasing economies of scale (β) increases the optimal quantity of shipment rapidly, increasing the coefficient of proportional cost (α) actually reduces the optimal amount of shipment. Holding β constant, increasing α amounts to an increase in the proportional cost and hence a relative erosion of the economies of scale. Therefore, the optimal amount of shipment declines. Intuitively, given a fixed ship, increasing labor cost forces a reduction in the amount of goods transported.

B. Impact of Production Technology

The capital stocks of the two countries fluctuate over time due to production shocks. The fluctuation is controlled by two parameters: σ and κ . The diffusion parameter σ determines the instantaneous impact of a production shock. The larger σ is, the larger the impact of the production shock. The drift parameter κ , on the other hand, controls the serial dependence of the capital stock. A negative κ is an indication of mean reversion while a positive κ is an indication of diversion. The more positive κ is, the faster the capital stock drifts away from its current position.

In making the optimal shipping decision, consumers are not only concerned with the cost of shipping, but also concerned with the dynamics of production shocks. For example, if one expects the capital stock will come back to balance quickly by itself due to the force of mean reversion, one may want to wait for the natural balancing event instead of incurring a cost in making a shipment. On the other hand, if one expects the capital stock imbalance to increase further in the future due to a divergent drift (positive κ), one may want to intervene early to stop that trend. Such considerations affect not only when to ship, but also how much to ship. Figure 2 depicts the impact of the two production parameters σ and κ on the optimal shipping decisions. Again, the left panels depict the impacts on λ_1 , which captures when to ship, and the right panels depict the impacts on ξ , the optimal shipment. The solid lines depict a general cost structure with both a constant proportion ($\alpha = 0.18$) and modest economies of scale ($\beta = 0.001$). The dashed lines depict the purely proportional cost case with zero economies of scale ($\beta = 0$).

The left panel in Figure 2 indicates that the direction of the impact of the production technology on the optimal boundary λ_1 remains the same with different cost structures. Nevertheless, the boundary λ_1 seems to become more sensitive to the technology parameters (σ, κ) with economies of scale in the cost structure (the solid lines) than without (the dashed lines). Overall, the optimal boundary λ_1 increases with the diffusion parameter σ but decreases with the drift parameter κ . As the diffusion parameter σ increases, one expects more variation in the capital stock and hence expects a higher chance that the capital stock comes back to balance by its own force. Given such expectations, consumers are willing to tolerate a higher imbalance (a larger λ_1) before starting intervention. On the other hand, a large positive κ implies that the higher the capital stock, the faster it increases. Such a force will certainly

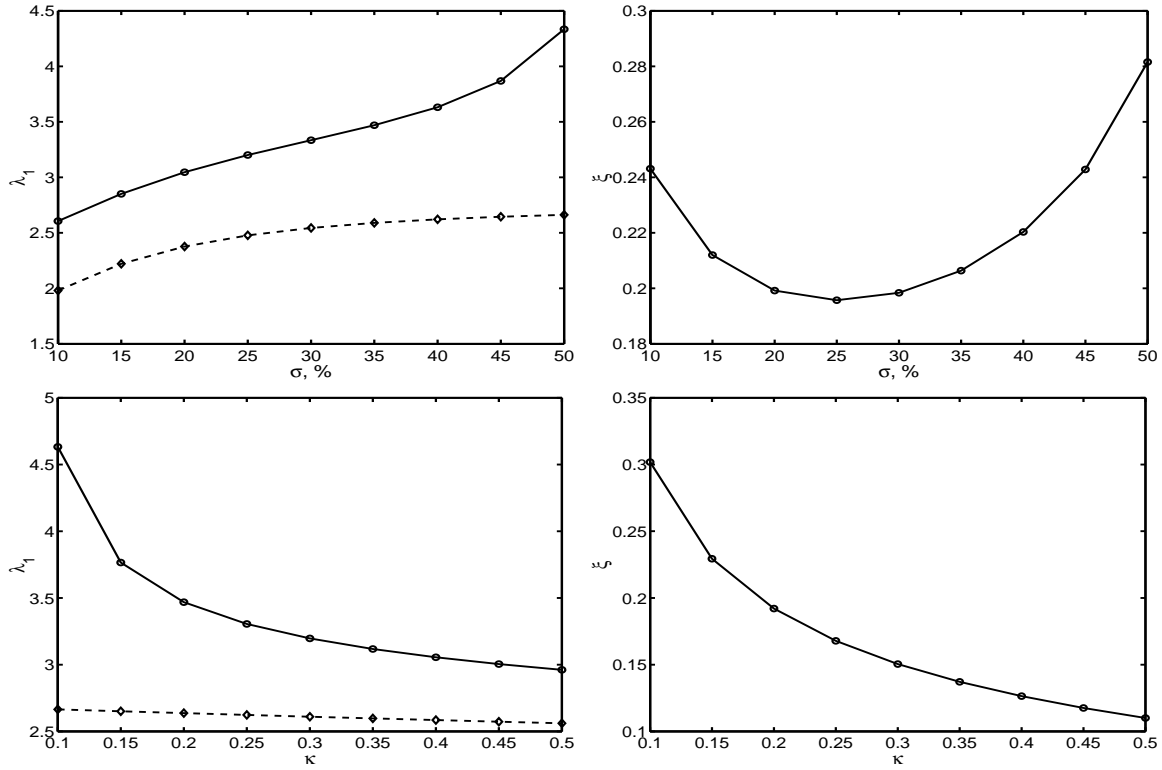


Figure 2. Impact of Production Shocks on Optimal Shipping Decisions

The left panels depict the optimal boundary at which the shipment takes place (λ_1) while the right panels depict the optimal shipment (ξ), both as a function of the diffusion parameter σ (first row) and the drift parameter κ (second row) of the capital stock dynamics. The circle-solid lines denote a general cost structure with model parameters: $\kappa = 0.11$, $\gamma = -1$, $\rho = 0.15$, $\alpha = 0.18$, and $\beta = 0.001$. The diamond-dashed lines denote the purely proportional cost counterpart with $\beta = 0$ and all other parameters the same.

drive a given imbalance to a further extent. Expecting future divergence in capital stock given a large positive κ , consumers take actions early and begin shipping at a smaller λ_1 .

The impact on the optimal shipping quantity is more complicated. In the purely proportional cost case, the optimal shipping quantity is infinitesimal, regardless of the production technology. In the presence of economies of scale in the cost structure, the optimal shipping quantity becomes a nontrivial decision and is sensitive to the variations in the production technology. Interestingly, as shown in the top right panel of Figure 2, the optimal shipment, ξ , exhibits a non-monotonic dependence on the diffusion parameter σ . The optimal shipping quantity first decreases and then increases with σ . Therefore, more than one force is at work. On the one hand, when the imbalance becomes larger, one in general needs to ship more to get back to balance. Such a force predicts an increase in ξ as σ increases. On the other hand, one may expect more frequent hitting of the boundary λ_1 and hence more frequent intervention. One might hence reduce the shipping quantity to reduce the total cost of transportation.

The impact of the drift parameter κ on the optimal shipping quantity ξ is simpler as ξ declines monotonically with increasing κ . Intuitively, expecting a divergent force in the capital stock movement, one makes small but frequent shipping to keep the capital stock close to balance.

C. Impact of Consumer Preference

Consumers discount both risk and time. This is captured, respectively, by consumers' relative risk aversion parameter $(1 - \gamma)$ and their time discount parameter ρ . Figure 3 depicts the dependence of the optimal shipping decisions on consumers' preference along both dimensions.

Both risk aversion and time discount increase consumers' preference for consumption smoothing and hence their desire to achieve and maintain a balanced account of capital stock. Therefore, the boundary λ_1 for optimal shipping decreases with both increasing relative risk aversion coefficient $(1 - \gamma)$ and increasing time discounting factor ρ . As the shipping frequency increases (λ_1 decreases), the shipping amount (ξ) usually declines. An exception occurs when the consumer has very high risk aversions. Then, the highly risk averse consumer not only wants to ship more frequently to maintain a narrower band $[\lambda_1, 1/\lambda_1]$, but also wants to ship as much as possible to achieve immediate reduction in the capital stock imbalance.

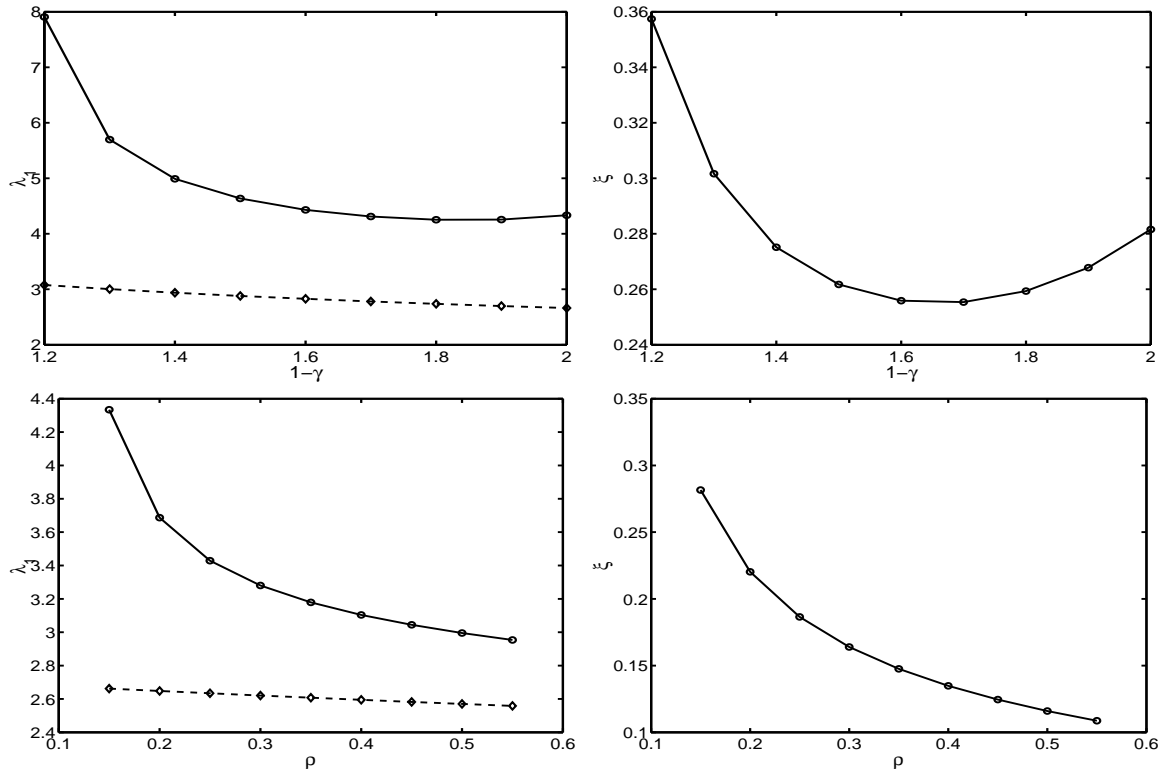


Figure 3. Impact of Consumer Preference on Optimal Shipping Decisions

The left panels depict the optimal boundary at which the shipment takes place (λ_1) while the right panels depict the optimal shipment (ξ), both as a function of the relative risk aversion ($1 - \gamma$) (first row) and the time discount factor ρ (second row). The circle-solid lines denote a general cost structure with model parameters: $\kappa = 0.11$, $\gamma = -1$, $\rho = 0.15$, $\alpha = 0.18$, and $\beta = 0.001$. The diamond-dashed lines denote the purely proportional cost counterpart with $\beta = 0$ and all other parameters the same.

To a consumer, optimal shipping is a decision of trade-offs between the benefit of risk diversification and the cost of intervention. The benefit of diversification drives the consumer to act more frequently and ship a larger amount to achieve a balance in capital stocks. The cost of shipment not only delays the consumer's action but also reduces the magnitude of the action. As the desire for diversification increases with increasing risk aversion, cost reduction becomes less important for the consumer. Therefore, a consumer with high relative risk aversion not only wants to intervene early at low λ_1 , but also wants to ship a larger amount (ξ) to achieve better consumption smoothing.

III. Optimal Consumption and the Capital Imbalance Dynamics

The optimal shipping decision and the optimal consumption decision interact to determine the dynamics of the capital stock imbalance in equilibrium. In particular, within the cone of no-shipment, the dynamics for $\omega \equiv \ln K/K^*$ can be derived from the two stochastic processes in (1) via Itô's lemma,

$$d\omega = \left[-\frac{c}{K} + \frac{c^*}{K^*} \right] dt + \sqrt{2}\sigma dz', \quad (20)$$

where $dz' = (dz - dz^*)/\sqrt{2}$ is a newly defined standardized Brownian motion. We observe that, within the no-trade zone, the physical imbalance ω has a constant instantaneous variance of $2\sigma^2$. The drift of the physical imbalance is determined by the difference in the consumption rates (the ratio of consumption to capital stock) between the two countries. Recall from (7) that the optimal consumption is determined by the marginal utility of the value function,

$$c = V_1^{\frac{1}{\gamma-1}}, \quad c^* = V_2^{\frac{1}{\gamma-1}}.$$

Apply the change of variables in (13), we have,

$$\frac{c}{K} = \left\| e^{I(-\omega)} I'(\omega) \right\|^{\frac{1}{\gamma-1}}, \quad \frac{c^*}{K^*} = \left\| e^{I(\omega)} I'(-\omega) \right\|^{\frac{1}{\gamma-1}}. \quad (21)$$

Obviously, the drift of the physical imbalance is a highly nonlinear function of the imbalance level ω . In Figure 4, we solve the transformed utility function and plot the drift of ω as a function of ω ,

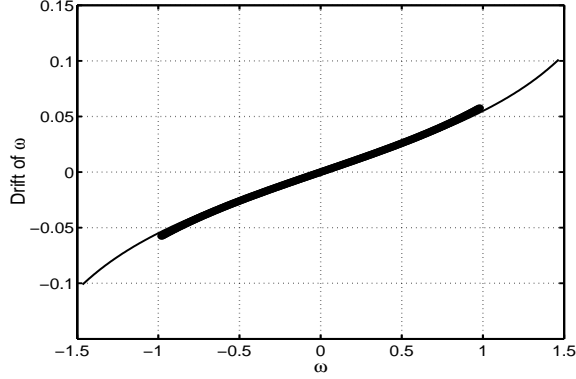


Figure 4. The drift of ω process

The solid line represents the drift function of ω under the following model parameters: $\kappa = 0.11, \sigma = 0.5, \rho = 0.15, \gamma = -1, \alpha = 0.18, \beta = 0.001$. The bold line denotes its limiting (purely proportional) case when we set $\beta = 0$.

for a given set of parameter values: $\kappa = 0.11, \sigma = 0.5, \rho = 0.15, \gamma = -1, \alpha = 0.18$. The economies of scale coefficient β is 0.001 for the solid line and zero for the bold line. In the purely proportional case ($\beta = 0$, bold line), the optimal shipping boundary is $\lambda_1 = 2.6621$ ($\omega_1 \equiv \ln \lambda_1 = 0.9791$) and the optimal shipping amount is infinitesimal. With an incorporation of a slight economies of scale ($\beta = 0.001$, solid line), the optimal shipping boundary moves outward to $\lambda_1 = 4.3342$ ($\omega_1 = 1.4665$) and the optimal shipping amount is $\xi = 28.16\%$ of the capital imbalance ($K - K^*$). The ex-shipping imbalance becomes $\lambda_2 = 1.9222$ ($\omega_2 = 0.6535$).

Note that under both cases, the drift increases with increasing capital imbalance. Thus, the dynamics of the capital imbalance is divergent and is constantly driven to one of the boundaries. This divergent feature is partially a result of consumption smoothing because the imbalance in optimal consumption is always less than the imbalance in capital stock between the two countries:

$$\begin{aligned} \frac{c}{c^*} &\leq \frac{K}{K^*}, & \text{when } K > K^* \\ \frac{c^*}{c} &\leq \frac{K^*}{K}, & \text{when } K^* > K. \end{aligned}$$

The impact of the cost structure on the capital imbalance dynamics is more vividly shown via the simulated sample paths of ω in Figure 5. The two panels share the same model parameters except that

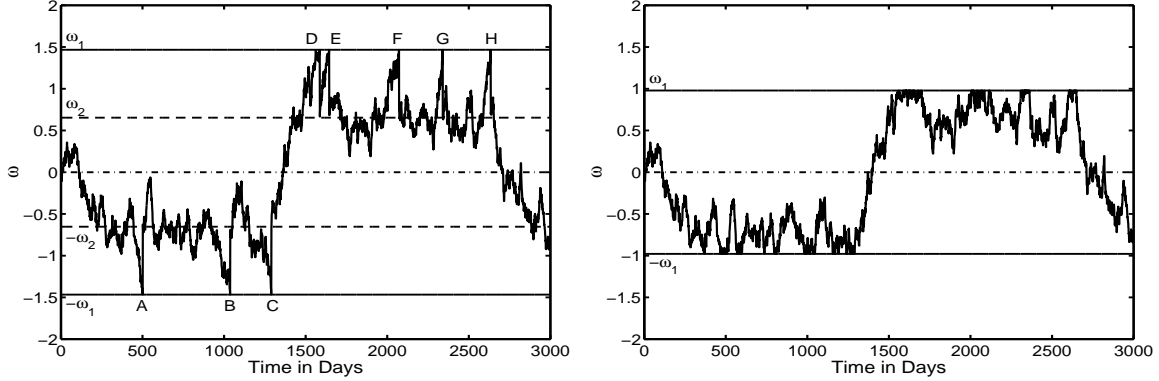


Figure 5. The Physical Imbalance Dynamics of Capital Stock, ω

The processes are simulated based on the model parameters: $\kappa = 0.11$, $\sigma = 0.5$, $\rho = 0.15$, $\gamma = -1$, $\alpha = 0.18$. The coefficient of economies β is set to 0.001 for the left panel and zero for the right panel. The solid lines at ω_1 and $-\omega_1$ depict the boundaries at which it is optimal to ship. The two dashed lines at ω_2 and $-\omega_2$ depict the ex-shipment imbalance. The solid and dashed lines overlap in absence of economies of scale (right panel). The dash-dotted line in the middle ($\omega = 0$) denotes the benchmark of exact capital balance. The points marked A, B, C, D, E, F, G, H in the left panel denote points where finite shipment occurs.

we set $\beta = 0.001$ for the left panel and $\beta = 0$ for the right panel. First, as just mentioned above, adding a very modest economies of scale component in the cost structure pushes the boundaries on capital imbalance further out before intervention occurs. But more importantly, the presence of economies of scale induces finite jumps in the physical imbalance process. Whenever either of the two optimal shipping boundaries (the two solid lines at ω_1 and $-\omega_1$) is reached, it is optimal to ship a finite proportion, $\xi = 0.2816$, so that the imbalance jumps inside the band. The ex-shipment boundaries are captured by the two dashed lines at ω_2 and $-\omega_2$. The points marked A, B, C, D, E, F, G, H in the left panel highlight the jumps in the imbalance process. In particular, as we have discussed in the previous section, the size of the jumps increases dramatically with the increase in the degree of economies of scale β . In contrast, in the absence of economies of scale in the cost structure (right panel), there are no jumps in the imbalance process. Whenever the boundary of ω_1 or $-\omega_1$ is reached, the physical imbalance is pushed back by an infinitesimal amount. This infinitesimal shipping, together with the divergent feature of the drift function, dictates that the physical imbalance stays at the boundary most of the time.

IV. The Dynamic Behavior of Real Exchange Rates

Given the optimal shipping decision and optimal consumption, one can determine the equilibrium real exchange rate by evaluating the marginal utilities. In this section, we analyze the equilibrium behavior of the real exchange rate and focus, in particular, on the impact of the cost structure on the dynamics of the real exchange rate. Specifically, we observe that the economies of scale component not only contributes in important ways to the slow convergence of the real exchange rate to parity, but also becomes indispensable in generating large short term real exchange rate volatility. We also observe distinctive behaviors of the real exchange rate implied by the model that have important implications for future empirical studies.

Specifically, let p denote the price of goods located at home relative to goods located abroad (the price of a unit of K in units of K^*). It can be regarded as the real exchange rate and is given by the ratio of the marginal utilities,

$$p = \frac{V_1(K, K^*)}{V_2(K, K^*)}. \quad (22)$$

Because of the homogeneity of V , the relative price p is a function of ω only:

$$p(\omega) = e^{-\omega} \frac{I'(\omega)}{I'(-\omega)}. \quad (23)$$

The Law of One Price (LOP) prevails when $p = 1$. Let $x(\omega) \equiv \ln p(\omega)$ denote the log real exchange rate, it is given by

$$x(\omega) \equiv \ln p(\omega) = -\omega + \ln I'(\omega) / I'(-\omega). \quad (24)$$

Equation (24) represents the real exchange rate as a function of the capital imbalance ω . Combine this with the dynamics of the physical imbalance in (20) and (21), we can derive the drift and diffusion function of the real exchange rate via Itô's lemma. As the functions are highly nonlinear and complicated, we analyze their properties through numerical examples.

A. Impact of the Transaction Cost Structure

In Table I, we report the values of the drift and the absolute magnitude of the instantaneous volatility of the real exchange rate under different real exchange rate levels: $\ln p = 0, -0.12, -0.24$. The left panel of Table I illustrates how the drift of the real exchange rate varies at different exchange rate levels and how it varies with the transaction cost structure in terms of α and β . First, we see that the drift is zero at parity ($x = 0$), but becomes positive when the exchange rate level is below parity, implying that the real exchange rate does have a force of mean-reverting back to parity. In other words, the purchasing power parity does hold in the long run. In particular, this mean-reverting force increases as the exchange rate level moves further away from the parity value. For example, consider the last row with $\alpha = 0.30$ and $\beta = 0.0015$. The drift increases from 0 to 0.014 and then to 0.032 as the real exchange rate moves from parity ($x = 0$) down to $x = -0.12$ and then to $x = -0.24$. Second, given the same α , the drift declines rapidly as one increases the magnitude of β . A small increase in β results in a large drop in the mean-reverting force, thus making the convergence to parity even slower. Third, usually increasing the proportional component of the cost α also reduces the drift and hence the speed of convergence to parity. But this is more so when economies of scale are absent or negligible. When the economies of scale coefficient $\beta = 0.0015$, the drift at $x = -0.12$ actually becomes non-monotonic. Fourth, when both α and β are small, the allowable variation of the real exchange rates is limited so that in some cases $x = -0.24$ is not reachable. This, however, is no longer the case when the cost is large, in terms of either α and/or β . A particularly interesting case occurs when $\alpha = 0.18$ and $\beta = 0.0010$, at which point the drift takes two possible values: 0.108 and 0.355. A similar case also occurs at $\alpha = 0.18$ and $\beta = 0.0015$.

The right panel of Table I illustrates how the instantaneous volatility (diffusion) of the real exchange rate varies across different exchange rate levels and with different cost structure parameters. Overall, the volatility declines as the exchange rate level moves away from parity. The decline is most dramatic when economies of scale (β) is small. An exception occurs at $\alpha = 0.18$ and $\beta = 0.0015$, when the volatility takes two values at $x = -0.24$ and one of them, 0.702, is significantly larger than the volatility levels at $x = 0$ or -0.12 . Holding α constant, at a given exchange rate level, a slight increase in β results in a large increase in volatility. Volatility normally also increases with increasing α , but the impact is not as dramatic as the case of β . In particular, when β is large, the impact of α becomes non-monotonic

in some instances. For example, at $\beta = 0.0015$, the volatility level actually decreases when α increases from 0.12 to 0.24. It is only after a further increase in α that the volatility starts to rise.

In short, both the drift and diffusion functions of the real exchange rate are highly nonlinear in the exchange rate levels and, in some cases, even indeterministic. Both the proportional component and the economies of scale component of the cost structure for goods transportation affect the real exchange rate dynamics in important ways. In particular, a slight increase in economies of scale significantly reduces the real exchange rate's speed of convergence to parity and increases its volatility at all levels. Thus, the model can readily generate the two stylized features of data: slow convergence to parity and high volatility. The impact of the proportional component is usually along the same direction as that of the economies of scale component, but less dramatic.

B. Indeterminacy in Real Exchange Rate Dynamics

From Table I, we have observed that at certain exchange rate levels, two possible values exist for both the drift and the diffusion of the exchange rate. To understand why this is happening, we graph the log exchange rate as a function of the physical imbalance ω in Figure 6. In the Figure, we plot the function for two cases: one with economies of scale ($\beta = 0.001$, solid line) and one without ($\beta = 0$, bold line). First, we observe that the addition of even a small economies of scale component ($\beta = 0.001$) moves the boundaries of the real exchange rate significantly further out from parity. Thus, the impact of β on the real exchange rate dynamics is big even if the magnitude of β is small. Even bigger, however, is the impact of β when the real exchange rate is away from parity. In particular, in between the before and after shipping boundaries $[\omega_1, \omega_2]$, the mapping from the real exchange rate to the physical imbalance is no longer unique. Two physical imbalance values correspond to one exchange rate level.

The key reason behind this non-unique mapping is the smooth pasting, which dictates that the marginal utilities are the same at the two boundaries ω_1 and ω_2 (and at also their negative counterparts). Thus, the real exchange rates are also the same at these two boundaries. The impact of such smooth pasting is analogous to the “honeymoon effect” described in Krugman (1991). As the physical imbalance increases, the deviation from parity increases. However, as the imbalance is near the outer boundary of ω_1 , consumers realize that the shipping of the capital stock is imminent. In anticipation

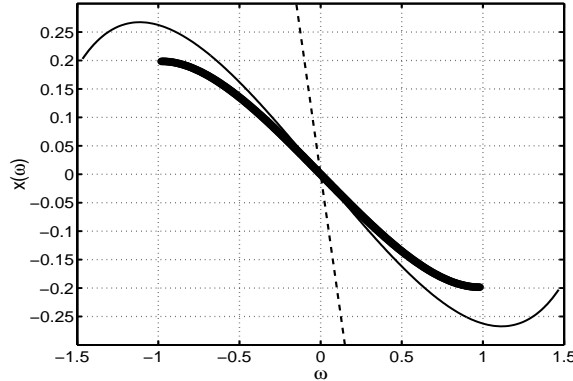


Figure 6. The Logarithm of the Real Exchange Rate

Lines depict the logarithm of the real exchange rate $x(\omega) \equiv \ln p(\omega)$ as a function of the physical imbalance of capital stock. The model parameters are $\kappa = 0.11$, $\sigma = 0.5$, $\rho = 0.15$, $\gamma = -1$; $\alpha = 0.18$. The coefficient of economies of scale is set to $\beta = 0.001$ for the solid line and to zero for the bold line. The dash-dotted line represents the autarky case with $x(\omega) = (\gamma - 1)\omega$.

of the shipment, the real exchange rate reverses direction and the deviation from the parity starts to decrease before reaching the boundary. Once the boundary is reached, a finite amount of capital stock is shipped from one country to the other; yet, no arbitrage dictates that the marginal utilities and hence the real exchange rate before and after the shipment are exactly the same. Thus, there are no jumps in the real exchange rate although there are jumps in the physical imbalance of the capital stock. Furthermore, the maximum deviation from purchasing power parity does not occur at the outer boundary of physical imbalance at which shipping occurs, nor does it occur at the ex-shipping boundary. Instead, the maximum is achieved between the two boundaries.

As the real exchange rate is a direct function of the physical imbalance of the capital stock, we can regard the physical imbalance ω as the state variable representing the underlying economy. Given the dynamics of the state variable, we can determine the dynamics of the real exchange rate. Nevertheless, due to the absence of one-to-one correspondence between the two, we cannot uniquely determine the state variable ω from the real exchange rate process. For example, at $x = -0.2028$, the state variable can either be at $\omega_1 = 1.4665$ or at $\omega_2 = 0.6536$.

This absence of one-to-one correspondence between the real exchange rate and the state variable has important implications for the exchange rate dynamics. In Figure 7, we depict the drift and diffusion functions of the log real exchange rate, $x(\omega)$. Again, we compare the case with economies of scale

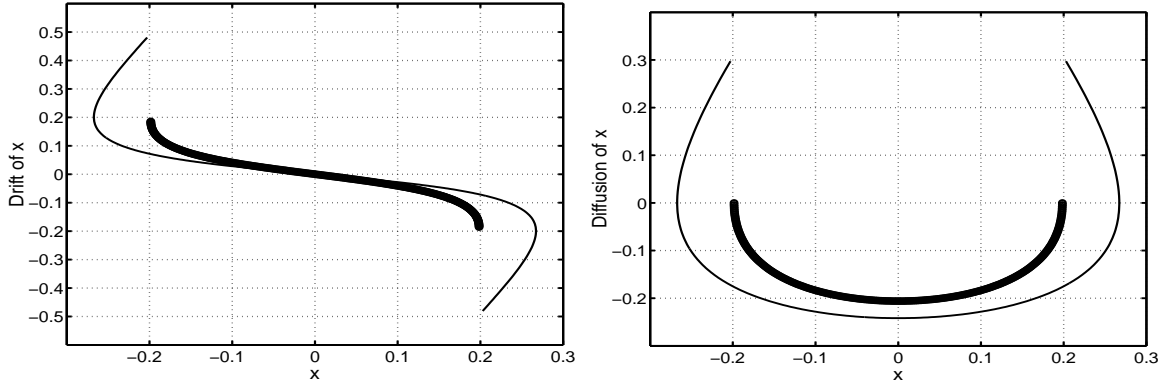


Figure 7. The Drift and Diffusion of the Log Real Exchange Rate $x(\omega)$

Lines depict the drift (left panel) and diffusion (right panel) of the log real exchange rate $x(\omega)$. The model parameters are: $\kappa = 0.11$, $\sigma = 0.5$, $\rho = 0.15$, $\gamma = -1$; $\alpha = 0.18$. The coefficient of economies of scale β is set to 0.001 for the solid line and zero for the bold line.

(solid line) and the case without economies of scale (bold line). We find that, holding everything else constant, incorporating a modest amount of economies of scale not only reduces the reverting speed to parity (the drift function is much flatter), but also increases the overall volatility level of the deviation (the diffusion function takes higher values at each exchange rate level).

What is most distinctive, however, is that, in the presence of the economies of scale, neither the drift nor the diffusion is completely determined by the current value of the exchange rate. For example, at $x = -0.24$, the drift of x can either be 0.108 or 0.355. Similarly, the diffusion can either be -0.124 or 0.176. Similar values are reported in Table I at $\alpha = 0.18$ and $\beta = 0.0010$, only with the sign differentiation for the volatility. As the sign of the diffusion indicates the different directions of the impact of a production shock, the different signs in the two diffusion values say that the same shock can either move the exchange rate positively or negatively, depending upon the current level of the capital imbalance.

Such indeterminacies have important implications for empirical studies. Time series analysis of the real exchange rate by itself is no longer capable of revealing the full information about the exchange rate dynamics. A more informative analysis may also need to include economic fundamentals that proxy for capital imbalance, such as trade deficit or capital flows. Dumas (1992) argues that a linear specification in estimating the real exchange rate dynamics is severely misspecified and hence may lead

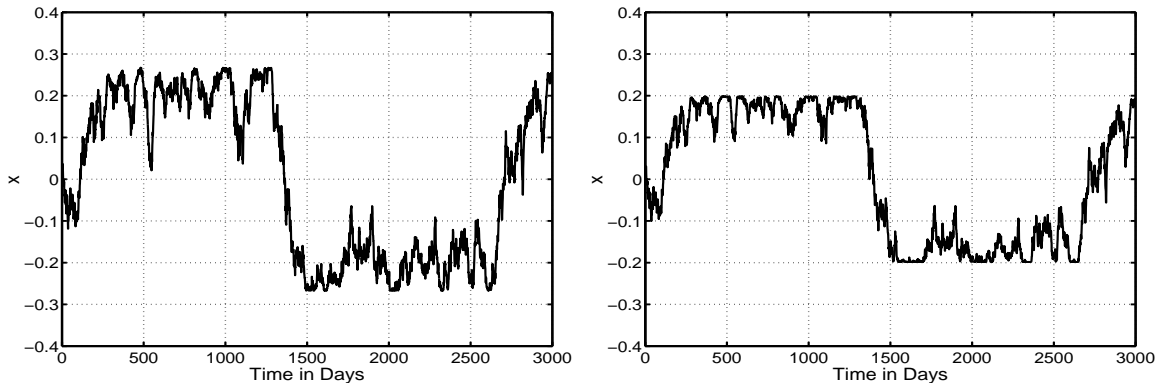


Figure 8. The Simulated Process of Real Exchange Rate, x

Model parameters are: $\kappa = 0.11, \sigma = 0.5, \rho = 0.15, \gamma = -1; \alpha = 0.18$. β is set to 0.001 for the left panel and zero for the right panel.

to erroneous conclusions. His modeling effort has motivated several empirical studies to explore non-linear specifications in testing the mean-reverting behavior of the real exchange rate. Our more realistic specification in the cost structure of goods transportation further indicates that a simple exogenous time series analysis of the real exchange rate may still lead to erroneous conclusions even if one considers nonlinearity and/or heteroskedasticity. The key issue is that the drift and diffusion functions of the real exchange rate are not completely determined by the real exchange rate levels. Our model argues for joint analysis of real exchange rates and economic fundamentals.

Another important observation from Figure 7 is that the instantaneous volatility of the real exchange rate is maximal at parity $x = 0$ in the purely proportional cost case. The instantaneous variance approaches zero monotonically as the deviation from parity increases and approaches its boundary. In contrast, in the presence of economies of scale, the maximum instantaneous variance does not occur at parity, nor does it happen at the maximum boundary, but at an intermediate level of deviation from parity. These differences transform directly to different behaviors in the exchange rate dynamics.

In Figure 8, we plot the simulated dynamics of the real exchange rate with (left panel) and without (right panel) economies of scale. The dynamics are derived from the simulated dynamics of the state variable ω in Figure 5. As expected, in the presence of economies of scale (left panel), the allowable band of deviation from parity is larger. But more importantly, in absence of economies of scale, the real exchange rate dwells at the boundary with small local variation. In contrast, in the presence of

economies of scale, the real exchange rate rarely dwells at the boundary, but nevertheless persistently stays away from parity. Furthermore, the deviations have much larger volatilities than in the purely proportional cost case.

V. Concluding Remarks

Real exchange rates exhibit enormous short term volatility and extremely slow convergence to parity. These features remain challenges to theoretical researchers and have been referred to as the Purchasing Power Parity (PPP) puzzle. Resolving the PPP puzzle constitutes a major focus of the international finance literature. In a survey paper, Rogoff (1996) suggests that international goods markets remain quite segmented, with large trading frictions due to transportation costs, tariff and non-tariff barriers, etc. This article constructs a real economy general equilibrium model that successfully explains both the short term real exchange rate volatility and its slow decay to parity. The model incorporates a general cost structure that allows economies of scale in goods transportation and captures the salient features of market segmentation in international trade. Under such a general cost structure, we propose an efficient procedure to simultaneously solve for the optimal shipping time and the optimal shipping quantity. Given the solution, we analyze the impact of the cost structure on the optimal shipping decision and on the equilibrium behavior of the real exchange rate dynamics. We find that small economies of scale can play a big role in generating high volatility and slow convergence to parity for real exchange rates.

We also find that, in the presence of economies of scale in the cost structure, one can no longer fully identify the real exchange rate dynamics from the exchange rate data alone. This indeterminacy in real exchange rate dynamics has important implications for future empirical research. Time series analysis of the real exchange rate data alone is not enough to fully reveal the exchange rate dynamics. Such data need to be complemented with time series data on economic fundamentals such as trade imbalance or capital flows. Motivated by earlier theoretical results from models with purely proportional cost, empirical research has focused on the role of potential nonlinearities in estimating the real exchange rate dynamics, e.g., Baum, Barkoulas, and Caglayan (2001), Lothian and Taylor (1996), Michael, Nobay, and Peel (1997), Sarantis (1999), and Taylor and Peel (2000). Our results further indicate that,

in the presence of economies of scale in the transportation cost, an exogenous time series analysis of the real exchange rate data alone may still lead to erroneous conclusions even if one considers nonlinearity and/or heteroskedasticity. The key issue is that the drift and diffusion functions of the real exchange rates are not completely determined by the real exchange rate levels. Our model argues for joint analysis of exchange rates and economic fundamentals.

Appendix A. Maximum Shipment

Lemma 1 *Irrespective of the cost structure, consumers never have the incentive to transfer more than the stock imbalance: $X_{\max} = |K - K^*|$.*

When the cost of goods transportation does not depend upon the amount of shipment, consumers have the incentive to transfer the maximal amount to achieve immediate stock balance. Let X_{\max} denote this maximal amount of the shipment when the total cost of transportation does not depend on the shipment. Let C denote this total cost. Assume that the home country has a larger capital stock than the foreign country, $K > K^*$, and decides to ship X_{\max} to the foreign country. We have

$$K - X_{\max} = K^* + X_{\max} - C. \quad (\text{A1})$$

That is, stock balance is achieved after the shipment. The equality in (A1) yields

$$X_{\max} = \frac{K - K^* + C}{2}, \quad (\text{A2})$$

which says that the amount of transfer increases with the total cost C .

However, for the shipment to be optimal, the total cost C cannot be too high. In particular, consumers in the foreign country have the incentive to receive a shipment only when they can gain from it. That is, the maximum total cost cannot be higher than the amount transferred, X . Thus,

$$C_{\max} = X_{\max}, \quad (\text{A3})$$

forms an upper bound for the total cost.

Combine the maximum cost equation in (A3) with the maximum shipment equation in (A2), we obtain

$$X_{\max} = K - K^*,$$

which represents the maximum possible amount of shipment in any cost structure.

When the foreign country has a larger capital stock, $K^* > K$, and decides to ship some of the stock to the home country, similar argument yields $X_{\max} = K^* - K$. Thus, in all cases, the maximum shipment is given by

$$X_{\max} = |K - K^*|.$$

All through the paper, we only consider the case of $K > K^*$ and determine the case of $K^* > K$ via symmetry.

Appendix B. Capital Imbalance Before and After Shipping

Let $\lambda_1 = K/K^*$ denote the capital imbalance before shipping and assume that an amount X is going to be shipped from home to abroad. Then, the capital stock of the home country reduces to $(K - X)$, and the capital stock of the foreign country becomes

$$K^* + X - TC = (1 + \beta)K^* + (1 - \alpha)X - \beta K.$$

Therefore, the after-shipping capital imbalance becomes

$$\begin{aligned} \lambda_2 &= \frac{K - X}{(1 + \beta)K^* + (1 - \alpha)X - \beta K} \\ &= \frac{\lambda_1 - X/K^*}{(1 + \beta) + (1 - \alpha)X/K^* - \beta\lambda_1} \\ &= \frac{\lambda_1 - \xi(\lambda_1 - 1)}{(1 + \beta) + (1 - \alpha)\xi(\lambda_1 - 1) - \beta\lambda_1}. \end{aligned}$$

The second line is obtained by dividing both the numerator and the denominator by K^* . The last line is obtained by plugging in the following equality:

$$\frac{X}{K^*} = \frac{X}{K - K^*} \frac{K - K^*}{K^*} = \xi(\lambda_1 - 1).$$

Appendix C. Optimal Shipping

Apply the change of variables to the first order condition in (12), we have

$$e^{(\eta-1)I(\omega_2)} = \frac{-I'(\omega_2)(1 + (1 - \alpha)\lambda_2) + (1 - \alpha)\lambda_2\gamma}{(\eta - 1)(T_1(\omega_2) - (1 - \alpha)\lambda_2 T_2(\omega_2))}, \quad (C4)$$

where

$$\begin{aligned} T_1 &= \zeta_1 \left((I')^2 + I'' - I' \right) + \zeta_2 \left(- (I')^2 + \gamma I' - I'' \right), \\ T_2 &= \zeta_1 \left(- (I')^2 + \gamma I' - I'' \right) + \zeta_2 \left((I' - \gamma)^2 + I'' + (I' - \gamma) \right). \end{aligned}$$

We drop the dependence of I', I'', ζ on ω_2 for clarity. We can also derive an expression for $e^{(\eta-1)I(\omega_2)}$ from the ordinary differential equation (14):

$$I'' = -(I')^2 + I'\gamma + \frac{1}{\sigma^2 \eta} e^{(\eta-1)I} [I'\zeta_1 + (\gamma - I')\zeta_2] + \delta. \quad (C5)$$

Combine the two equations (C4) and (C5), we cancel out the $I(\omega_2)$ term and obtain a quadratic function of $I''(\omega_2)$ in terms of $I'(\omega_2)$:

$$A (I'')^2 + BI'' + C = 0, \quad (C6)$$

where the coefficients $[A, B, C]$ are functions of $I'(\omega_2)$ and model parameters. They are given by

$$\begin{aligned} A &= (\zeta_1 - \zeta_2)(1 + (1 - \alpha)\lambda_2) \\ B &= G + EA; \\ C &= EG - F. \end{aligned}$$

with

$$\begin{aligned} G &= I' [\zeta_1 (I' - 1) + \zeta_2 (-I' + \gamma)] - (1 - \alpha)\lambda_2 (I' - \gamma) [-\zeta_1 I' + \zeta_2 ((I' - \gamma) + 1)], \\ E &= (I')^2 - I'\gamma - \delta, \\ F &= \frac{(I' + (1 - \alpha)\lambda_2 (I' - \gamma)) [-\zeta_1 I' + \zeta_2 (I' - \gamma)]}{\eta(\eta - 1)\sigma^2}. \end{aligned}$$

Given that the discriminant $B^2 - 4AC > 0$, the quadratic equation has two real roots. We choose the root that ensures convergence for the ordinary differential equation. In particular, a geometric analysis indicates that the convergent root is

$$I''(\omega_2) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \quad (C7)$$

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Table I
Drift and Diffusion of Real Exchange Rates

Entries report the values of the drift and instantaneous volatility of the real exchange rates under fixed real exchange rate levels: $x \equiv \ln p = 0, -0.12, -0.24$. The values are computed under model parameters are: $\kappa = 0.11, \sigma = 0.5, \gamma = -1, \rho = 0.15$. The values for the cost coefficients (α, β) are given in the table. Entries with “Out of Bounds” notes imply that the real exchange rate cannot deviate that much from parity under the given model parameter sets.

x	Drift			Volatility		
	0	-0.12	-0.24	0	-0.12	-0.24
<u>$\alpha = 0.12$</u>						
$\beta = 0.0000$	0.000	0.113	Out of Bounds	0.161	0.064	Out of Bounds
$\beta = 0.0005$	0.000	0.079	Out of Bounds	0.177	0.117	Out of Bounds
$\beta = 0.0010$	0.000	0.044	Out of Bounds	0.220	0.193	Out of Bounds
$\beta = 0.0015$	0.000	0.010	0.022	0.359	0.354	0.339
<u>$\alpha = 0.18$</u>						
$\beta = 0.0000$	0.000	0.052	Out of Bounds	0.206	0.172	Out of Bounds
$\beta = 0.0005$	0.000	0.044	Out of Bounds	0.218	0.191	Out of Bounds
$\beta = 0.0010$	0.000	0.034	0.108/0.355	0.242	0.222	0.124/0.176
$\beta = 0.0015$	0.000	0.016	0.038/1.042	0.316	0.308	0.280/0.702
<u>$\alpha = 0.24$</u>						
$\beta = 0.0000$	0.000	0.033	0.100	0.245	0.226	0.137
$\beta = 0.0005$	0.000	0.030	0.082	0.256	0.240	0.170
$\beta = 0.0010$	0.000	0.025	0.063	0.274	0.260	0.210
$\beta = 0.0015$	0.000	0.018	0.041	0.309	0.301	0.270
<u>$\alpha = 0.30$</u>						
$\beta = 0.0000$	0.000	0.023	0.058	0.281	0.269	0.223
$\beta = 0.0005$	0.000	0.021	0.051	0.291	0.280	0.241
$\beta = 0.0010$	0.000	0.018	0.043	0.306	0.296	0.265
$\beta = 0.0015$	0.000	0.014	0.032	0.332	0.325	0.302